ZIMBABWE

Ministry of Primary and Secondary Education

Mathematics Module Level II Volume 1

Lifelong and Continuing Education

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Introduction

The last thirty years have seen resurgence in Open Distance learning as a pedagogical approach and this trend is envisaged to continue. The knowledge-based society that we live in has enabled learning to take place anywhere or everywhere. The concept of a classroom without walls continues to grow in Zimbabwe. Due to the demand for open distance learning, the Ministry of Primary and Secondary Education has revamped its non-formal education department to embed distance learning as a tool for learning in order to address the learning needs of the growing numbers of out of school learners or school drop outs that cannot access formal education systems. The module is written in a simple manner with lots of friendly and interactive activities to make learning interesting and easier for the out of school learners. The module develops critical thinking skills, problem solving skills among other 21st Century skills.

It is the Ministry's hope that out of school learners are going to take advantage of this module and benefit immensely in advancing their learning endeavours.

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How to use this module

As you start this journey of acquiring a qualification in Ordinary Level Mathematics through open distance learning, it is critical that you understand the need to manage your study time and balance it with your day-to-day activities. This module will provide you with the basic material to assist you towards your public examinations in Mathematics.

This module has been subdivided into two volumes, that is, Volume 1 Volume 2. You are advised to study Volume 1 first before going to Volume 2.

Wish you the best!

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UNIT 1 - REAL NUMBERS

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1.1 INTRODUCTION

This unit seeks to identify the different types of real numbers with special reference to rational and irrational numbers. This unit is also going to look at how to add, subtract, multiply and divide real numbers that you meet in your daily situations

OBJECTIVES

After going through this unit, you should be able to:

- distinguish between rational and irrational numbers
- perform operations involving real numbers
- solve problems involving distance and area using scale
- convert a number in one base to the other
- solve equations involving number bases
- identify missing numbers in a sequence

KEY TERMS

Rational numbers: –these are numbers that can be written as fractions for example $\frac{1}{4}, \frac{7}{5}, 0{,}75 \text{ or } \frac{a}{b}$, .

Irrational numbers :– these are numbers that cannot be written as exact fractions, for example $\sqrt{2}, \sqrt{3} \text{ or } \pi$

Standard form:– this involves writing a number in the form $A \times 10^n$where A is a number between 1 and 10 while **n** is a positive or negative whole number(integer).?

Binary:– means base two.

Denary:– means base ten.

Sequence :– a set of numbers with a well-defined order or pattern, for example{2;4;6;8}or {1; 4; 9}

TIME:- you should not spend more than 10 hours in this unit.

STUDY SKILLS

The key skill to mastery of mathematical concepts is practice. You need to work out as many problems related to the topic on Real Numbers as possible for you to grasp all the concepts in this unit.

1.2 NUMBER CONCEPTS AND OPERATIONS

What is a number?

A number is a way to represent quantity. Numbers are not something that you can touch or hold, because they are not physical. When we look at the world around us, we quantify it using numbers. For example, we may say; How many learners?

How much money? How much distance? These are some of the questions which can only be answered using numbers.

A number can be written in many different ways and it is always advisable to choose the most appropriate way of writing the number. For example, "a quarter" may be spoken aloud or written in words. It can also be written as a fraction or as a decimal number 0,25.

Real numbers

These are numbers found on a number line, both negative and positive numbers Now that we have learnt about what a number is, we can look at what real numbers are in greater detail. The following are examples of real numbers and it is seen that each number is written in a different way.

$\sqrt{5}$ 2.45434 $\frac{7}{3}$ π –58 $-\frac{19}{8}$ – 56.8

Now, from the list below identify real numbers in the space below

$1\frac{9}{5}$, 6k, $-\frac{1}{2}$, –9, 3x, 4z, 2π, 7, $\frac{x}{3}$

Hint: If you have chosen numbers without letters you got it right.
Depending on how the real number is written, it can be further labelled as either rational, irrational, integer or natural.

Natural Numbers

The first type of numbers that are learnt about are the numbers that are used for counting. These numbers are called natural numbers and are the simplest numbers in mathematics:

1, 2, 3, 4, . . .

Mathematicians use the symbol $\mathbb{N}$ to mean the set of all natural numbers. These are also called positive whole numbers. The natural numbers are a subset or part of the real numbers since every natural number is also a real number.

Integers

Integers are positive and negative whole numbers :
{. . . − 4,−3,−2,−1, 0, 1, 2, 3, 4 . . .}
Mathematicians use the symbol $\mathbb{Z}$ to mean the set of all integers. The integers are a subset of real numbers, since every integer is a real number.

Rational Numbers

Natural numbers and integers are only able to describe quantities that are whole or complete. For example you can have 4 apples, but what happens when you divide one apple into 4 equal parts and share it among your friends? Then it is not a whole apple anymore and a different type of number is needed to describe the apples. This type of number is known as a rational number.

A rational number is any number which can be written as $\frac{\mathbf{a}}{\mathbf{b}}$, where a and b are integers and b≠0.

The following are examples of rational numbers:

$$\{\frac{7}{3}; \quad \frac{-1}{2}; \quad \frac{40}{10}; \quad \frac{5}{20}; \quad \frac{10}{1}; \quad 11\}$$

Now, from the list below identify rational numbers in the space below

$\{5; \quad \frac{-5}{3}; \quad -9; \quad \frac{17}{19}; \quad \pi; \quad \frac{10}{1}\}$

Mathematicians use the symbol $\mathbb{Q}$ to mean the set of all rational numbers. A rational number may be a proper or improper fraction.

Irrational Numbers

An irrational number is any real number that cannot be expressed as an exact fraction. Irrational numbers when expressed as decimals, they can never be fully written out as they have an infinite number of decimal places which never fall into a repeating pattern. Pii (π) is a unique example of an irrational number. You may argue that π can be written as $\frac{22}{7}$, this value of π is merely an approximate value of π that is why we tend to classify π under irrational numbers.

For example,

$\sqrt{2} = 1{,}41421356 \ldots$

$\pi = 3{,}14159265 \ldots$

Activity (*1.1*) Rational and Irrational numbers

Questions

(a) List any 5 examples of rational numbers between 2 and 3

(b) From the list of numbers below identify rational numbers:

0.5; 3.7; 2; $\sqrt{3}$; 6.25; $\sqrt{\frac{5}{2}}$; π;

$3\frac{1}{4}$; $\sqrt{7}$

Answers

a) Examples of rational numbers between 2 and 3 are:

{2,25 2,65; $2\frac{1}{3}$; 2,75; 2,95}

b) Rational numbers: { 0,5; 3,7; 2; 6,25; $3\frac{1}{4}$ }

FRACTIONS

From Level 1 you are aware that fractions can be written as proper, improper and mixed fractions. Just like any other type of real numbers you should be able to add, subtract, multiply and divide fractions.

Addition and subtraction

When adding and subtracting fractions, put the fractions under a common denominator. If the fractions are mixed change them to improper fractions first. Now, look at the examples given below.

Worked Example [1]

Questions

Evaluate each of the following and write the answers as a fraction in its lowest terms.

a) $\frac{3}{5}+\frac{2}{7}$ b) $\frac{3}{5}-\frac{2}{7}$ c) $6\frac{1}{5}-2\frac{3}{4}+1\frac{1}{6}$

Solutions

a) $\frac{3}{5}+\frac{2}{7}=\frac{3\times7+2\times5}{35}=\frac{21+10}{35}=\frac{31}{35}$

b) $\frac{3}{5}-\frac{2}{7}=\frac{3\times7-2\times5}{35}=\frac{21-10}{35}=\frac{11}{35}$

c) $6\frac{1}{5}-2\frac{3}{4}+1\frac{1}{6}=\frac{31}{5}-\frac{11}{4}+\frac{7}{6}=\frac{31\times12-11\times15+7\times10}{60}=\frac{372-165+70}{60}=\frac{277}{60}=4\frac{37}{60}$

TIP: You are advised to change mixed numbers to improper fractions first.

Multiplication

When multiplying fractions, you start by multiplying the numerators on their own, do the same with the denominators and simplify to its lowest terms.

Worked Example [2]

Questions

Evaluate the following

a) $\frac{3}{5}\times\frac{2}{7}$

b) $\frac{7}{9}\times\frac{3}{5}$

Solutions

a) $\frac{3}{5}\times\frac{2}{7}=\frac{3\times2}{5\times7}=\frac{6}{35}$

b) $\frac{7}{9}\times\frac{3}{5}=\frac{7\times3}{9\times5}=\frac{21}{45}=\frac{7}{15}$

Division

Dividing fractions is the same as multiplying by the reciprocal. The reciprocal of a number is obtained when 1 is divided by the number.

For example, the reciprocal of 4 is $\frac{1}{4}$, and the reciprocal of $\frac{3}{5}$ is $\frac{5}{3}$.

Worked Example [3]

Questions

1) Find the reciprocal of the following

a) 6 b) $\frac{5}{8}$ c) $\frac{3}{4}$ d) $1\frac{1}{2}$

2) Evaluate the following

a) $\frac{5}{8} \div \frac{3}{7}$ b) $\frac{3}{5} \div \frac{2}{3}$

Solutions

1) a) $\frac{1}{6}$ b) $\frac{8}{5}$ c) $\frac{4}{3}$ d) $\frac{2}{3}$

2) a) $\frac{5}{8} \div \frac{3}{7} = \frac{5}{8} \times \frac{7}{3} = \frac{35}{24} = 1\frac{11}{25}$ b) $\frac{3}{5} \div \frac{2}{3} = \frac{3}{5} \times \frac{3}{2} = \frac{9}{10}$

TIP: When dividing fractions, you change the division sign to a multiplication sign then invert the second fraction.

Now that we have gone through some examples, attempt the questions in the following Activity.

Activity (*1.2*) Fractions

Questions

Simplify the following.

(a) $4\frac{3}{5} - 1\frac{5}{8}$

(b) $3\frac{1}{7} + 1\frac{2}{3}$

(c) $9\frac{1}{3} + 5\frac{3}{4} - 6\frac{1}{2}$

(d) $4\frac{1}{3} \times 2\frac{2}{5}$

(e) $1\frac{1}{9} \div 1\frac{1}{3} \times 3\frac{3}{5}$

Answers

(a) $2\frac{39}{40}$ (b) $4\frac{17}{21}$ (c) $8\frac{7}{12}$ (d) $10\frac{2}{5}$ (e) 3

DECIMAL NUMBERS

Decimal numbers are numbers with a comma. Just like any other type of real numbers you should be able to add, subtract, multiply and divide decimal numbers.

Addition and subtraction of decimal numbers

When adding and subtracting decimal numbers, you arrange the numbers in such a way that the commas are in a straight line downwards. Check this in the following worked example.

Worked Example [4]

Questions

Evaluate the following

a) $20.05 + 0.5$

b) $88.6 - 8.6$

c) $43{,}012 + 3{,}57 - 6{,}078$

Solutions

$$\begin{array}{r} \text{a) } 20{,}05 \\ +0{,}50 \\ \hline 20{,}55 \\ \hline \end{array} \qquad \begin{array}{r} \text{b) } 88{,}60 \\ -\ 8{,}65 \\ \hline 79.95 \\ \hline \end{array} \qquad \begin{array}{r} \text{c) } 43{,}012 \\ +\ 3{,}570 \\ \hline 46.582 \\ -\ 6{,}078 \\ \hline 40{,}504 \\ \hline \end{array}$$

Multiplication of decimal numbers

When multiplying decimal numbers, ignore the comma, multiply as whole numbers and put the comma in the final answer as determined by the sum of decimal places in the numbers being multiplied.

Worked Example [5]

Questions

Evaluate the following

a) $6{,}05 \times 0{,}25$ b) $4{,}15 \times 0{,}013$

Solutions

a) $6{,}05 \times 0{,}25 = 1{,}5125$ Working ($605 \times 25 = 15125$), then count the decimal places

b) $4{,}15 \times 0{,}013 = 0{,}05395$ Working ($415 \times 13 = 5395$)

Division of decimal numbers

When dividing decimal numbers make the denominator a whole number by multiplying by 10, 100 or 1000 depending on the number of decimal places in the denominator.

Worked Example [6]

Questions

Evaluate the following

a) $0{,}25 \div 0{,}125$ b) $0{,}5 \div 0{,}05$ c) $0{,}0216 \div 0{,}02$

Solutions

a) $0{,}25 \div 0{,}125$

$= \frac{0{,}25}{0{,}125} \times \frac{1000}{1000} = \frac{250}{125} = 2$

b) $0{,}05 \div 0{,}5$

$= \frac{0{,}05}{0{,}5} \times \frac{10}{10} = \frac{0{,}5}{5} = 0{,}1$

c) $0{,}0216 \div 0{,}02$

$= \frac{0{,}0216}{0{,}02} \times \frac{100}{100} = \frac{2{,}16}{2} = 1{,}08$

Now that we have gone through some examples, attempt the questions on the following activity.

Activity (*1.3*) Decimal numbers

Questions

1) Simplify the following

   a) $3.73 + 93.68 - 7.30$

   b) 6.32×3.4

   c) $0.0318 \div 0.003$

   d) $(0.3)^3 \times 0.25 \div (0.04)^2$

2) During a cyclone 0,6 of the cattle in a village were destroyed and only 240 cattle survived. How many cattle were in the village before the cyclone?

Answers

1) (a) 90.11 (b) 21.488 (c) 10.6 (d) 4.21875

2) 600

PERCENTAGES

Percentage refers to the amount, number or rate of something, regarded as part of a total of 100. Percentages are fractions whose denominator is 100 for instance $\frac{85}{100}$ can be written as 85%.Let us go through the following examples on how percentages are calculated.

Worked Example [7]

Questions

1) Out of 50 learners in a class, 30 are girls and 20 are boys.
   a) What percentage of learners are girls?
   b) What percentage of learners are boys?
2) Convert the following percentages to fractions and decimals
   a) 23% b) 5%
3) Convert the following to percentages
   a) $\frac{5}{8}$ b) 0,35

Solutions

1) a) $\frac{30}{50} \times 100$

   $= \frac{3}{5} \times 100$

   $= 60\%$

   b) $\frac{20}{50} \times 100$

   $= \frac{2}{5} \times 100$

   $= 40\%$

2) a) $\frac{23}{100} = 0{,}23$

   c) $\frac{5}{100} = 0{,}05$

3) a) $\frac{5}{8} \times 100$

   $= 62{,}5\%$

   b) $0{,}35 = \frac{35}{100} = 35\%$

Calculating a percentage of a quantity

You might need to give 10% of your money to your friend. How then can you calculate the amount of money you will give to your friend? The following examples will illustrate how this is done.

Worked Example [8]

Question

Evaluate the following

a) 10% of \$80

b) 25% of 200 metres

Solutions

a) $\frac{10}{100} \times \$80$

$= \$8$

b) $\frac{25}{100} \times 200$

$= 50$ metres

Expressing one quantity as a percentage of another

When expressing one quantity as a percentage of another, you first write the first quantity as a fraction of the second and then multiply by 100. Let's now consider the following examples.

Worked Example [9]

Questions

Express the results of Learner A and Learner B as a percentage

a) Learner A got 15 out of 60

b) Learner B got 50 out of 60

Solutions

a) $\frac{15}{60} \times 100$

$= 25\%$

b) $\frac{50}{60} \times 100$

$= 83.3\%$

Percentage Increase and Decreases

Suppose you are working in a company earning $10 000 per month and your salary is to be increased by 20%. Use the space below to calculate your new salary.

Worked Example [10]

Questions

1) A shop assistant has a salary of $10 000 per month. If her salary is increased by 20%, calculate
   a) The amount extra she will receive a month
   b) Her new monthly salary
2) A garage increases the price of fuel by 40%. If the original price was $9, calculate its new price.
3) A rural shop is having a sale. They are selling a pair of shoes costing $75 at a 25% discount. Calculate the sale price of the pair of shoes.

Solutions

1) a) Increase = 20% of $10 000

$\frac{20}{100} \times 10\ 000$

= $2 000

b) New Salary = Old salary + Increase

=$10 000 + $2 000

=$12 000

2) The original price represents 100%, therefore the new price represents 140%

New price = 140% of \$9

$= \frac{140}{100} \times \$9$

$= 1.4 \times \$9$

= \$12.60

3) The old price represents 100%, therefore the new price can be represented as

(100 – 25) % = 75%

75% of \$75 = $\frac{75}{100} \times \$75$

= \$56.25

Now that we have gone through some examples of calculations involving percentages, attempt the questions on the following activity.

Activity (*1.4*) Percentage increase and decrease

Questions

The marked price of a shirt in a shop is \$75.A discount of 25% is given for cash payment.

a) If Mr Kamota paid cash. How much did he pay for the shirt?

b) Mrs Pindani a flea market vendor bought 12 such shirts for cash. She went on to sell the shirts at \$85 each. Calculate her profit.

c) The marked price of \$75 of the shirt had risen by 20% from that of last year. Calculate the price of the shirt last year.

Answers

(a) \$56,25 (b) \$345 (c) \$62.50

1.3 RATIOS, RATES AND PROPORTIONS

Ratios

Ratios are used to compare related quantities. The simplest form of writing a ratio is a:b:c where a, b and c are whole numbers. Ratios can also be given as fractions for example 1: $\frac{2}{3}$ and this has to be simplified by multiplying by the common denominator, which is 3 in this case to give 3:2.

Worked Example [11]

Questions

1) When cooking rice, usually 1 cup of rice is mixed with 3 cups of water. What is the ratio of rice to water?
2) Ages of three family members are given as 18 years, 12 years and 3,6 years. Represent their ages as a ratio.
3) A farmer has 80hectares of land. He would like to distribute the land in the ratio 3:2:5 for cultivating beans, maize and tobacco respectively. Calculate the size of land for each of the crops

Solutions

1) The ratio is 1:3.
2) The ages can be written as a ratio as
   - 18years:12years:3,6years
   - Which can be simplified to 18:12:3,6.

     The decimal point can be removed by multiplying each number by 10

     This gives 180:120:36which can be further simplified to by dividing by 12 to give 15:10:3

3) Total ratio = 3+2+5 = 10

Beans = $\frac{3}{10} \times \frac{80}{1}$ = 24 hectares

Maize = $\frac{2}{10} \times \frac{80}{1}$ = 16 hectares

Tobacco = $\frac{5}{10} \times \frac{80}{1}$ = 40 hectares

Now that we have gone through some few examples, attempt the following questions on the following activity.

Activity (*1.5*) Ratios

Questions

1) Divide the following quantities in the given ratio
   a) $100 in the ratio 5:3
   b) 50km in the ratio 3:2
2) A factory produces cars in red, blue, white and green in the ratio 7:5:3:1. Out of a production of 32 000 cars, how many cars are
   a) Red
   b) White

Answers

1) a) (i) $62.5
   (ii) $37.5
   b) (i) 30km
   (ii) 20km
2) a) 14 000 red cars
   b) 6000 cars are white

Rates

Rates are used to compare quantities that are not related for example distance and time, number of papers produced by a photocopying machine over a period of time.

Worked Example [11]

Questions

1) A faulty Range Rover consumes 2 litres of petrol over a distance of 10km. Calculate the amount of petrol consumed over a distance of 200km.
2) A grinding mill produces 20 bags of mealie-meal over a period of 2 hours. Calculate the number of bags produced in 3 hours.

Solutions

1) You may work the question like this:

   10km = 2litres

   200km = x litres

   Number of litres = $\frac{200\text{km}}{10\text{km}} \times \frac{2\text{litres}}{1}$ = 40litres
2) 20 bags = 2 hrs

   x bags = 3 hrs

   Number of bags = $\frac{3\text{ hrs}}{2\text{ hrs}} \times \frac{20\text{ bags}}{1} = 30\text{ bags}$

Proportion

Proportion is used to compare quantities or to share quantities.

Worked Example [13]

Questions

Three boys Mac, Tinashe and Tom are aged 12, 14 and 20 respectively. They would like to share 69 oranges in the ratio of their ages. Calculate how many oranges each one of them is going to get.

Solution

Ratio is 12:14:20 reduced to 6:7:10 by dividing each number by 2

Total share=6+7+10=23

- Mac's share = $\frac{6}{23} \times \frac{69}{1}$ = 18 oranges

- Tinashe's share = $\frac{7}{23} \times \frac{69}{1}$ = 21 oranges
- Tom's share = $\frac{10}{23} \times \frac{69}{1}$ = 30 oranges

1.4 IRRATIONAL NUMBERS

Numbers that cannot be written as fractions or those numbers that do not have exact square roots are called surds for example $\sqrt{3}$, $\sqrt{5}$, $3\sqrt{7}$. Surds are a subset of irrational numbers. Surds can be combined for example, in the case of $\sqrt{20}$ can be written as $\sqrt{4 \times 5} = \sqrt{4} \times \sqrt{5} = 2\sqrt{5}$.

Now, for the list below identify surds in the space below

$\sqrt{2}$, $3\sqrt{11}$, $\sqrt{16}$, $\sqrt{1\frac{9}{16}}$, $\sqrt{12}$, 5

Hint: Let's hope you have noted that numbers without exact square roots are your answer

Surds can be manipulated and simplified according to a number of rules.

1. $\sqrt{a} \times \sqrt{a} = a$
2. $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$
3. $\frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}$

Simplifying surds

The square root of some numbers which are not prime numbers can be simplified. The square root of a number cannot be simplified when it is in its basic form.

Worked Example [14]

Question

Simplify the following surds

a) $\sqrt{20}$ b) $\sqrt{27}$ c) $\sqrt{12}$

Solution

a) $\sqrt{20} = \sqrt{4} \times \sqrt{5} =$
$2\sqrt{5}$

b) $\sqrt{27} = \sqrt{9} \times \sqrt{3} =$
$3\sqrt{3}$

c) $\sqrt{12} = \sqrt{4} \times \sqrt{3} =$
$2\sqrt{3}$

Addition and Subtraction

In order for you to add or subtract irrational numbers make sure numbers under square root are the same.

Worked Example [15]

Questions

Evaluate the following where possible

a) $4\sqrt{3} + 2\sqrt{3}$ b) $4\sqrt{2} + 3\sqrt{5}$ c) $4\sqrt{3} - 2\sqrt{3}$ d) $\sqrt{3} + \sqrt{12}$

Solutions

a) $4\sqrt{3} + 2\sqrt{3}$ This can be taken as **4a** + **2a** = **6a**, thus $\sqrt{3}$ is taken as an unknown (**a**)
$= 6\sqrt{3}$

b) $4\sqrt{2} + 3\sqrt{5}$ cannot be added because the numbers below the √ are not the same, this can be taken as 4c+3d

c) $4\sqrt{3} - 2\sqrt{3}$
$= 2\sqrt{3}$

d) $\sqrt{3} + \sqrt{12}$

$= \sqrt{3} + \sqrt{4} \times \sqrt{3}$

$= \sqrt{3} + 2\times \sqrt{3}$

$= \sqrt{3} + 2\sqrt{3}$ Note that $\sqrt{3}$ is the same as $1\sqrt{3}$

$= 3\sqrt{3}$

Multiplication

Just like any other type of numbers, surds can be multiplied.

Worked Example [16]

Questions

Evaluate the following

a) $\sqrt{3} \times \sqrt{2}$ b) $\sqrt{3} \times \sqrt{3}$ c) $3\sqrt{2} \times 4\sqrt{5}$

Solutions

a) $\sqrt{3} \times \sqrt{2} = \sqrt{3 \times 2} = \sqrt{6}$

b) $\sqrt{3} \times \sqrt{3} = \sqrt{3 \times 3} = \sqrt{9} = 3$

c) $3\sqrt{2} \times 4\sqrt{5} = 3 \times 4 \times \sqrt{2} \times \sqrt{5} = 12\sqrt{10}$

Division

When dividing surds, multiply both the numerator and the denominator by the denominator.

This process of dividing surds is called rationalisation of the denominator.

Worked Example [17]

Question

Rationalise the following

a) $\frac{2}{\sqrt{3}}$ b) $\frac{3}{\sqrt{5}}$

Solutions

a) $\frac{2}{\sqrt{3}}$ multiply both numerator and denominator by $\sqrt{3}$.

$$\frac{2}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}}$$

$$= \frac{2\sqrt{3}}{\sqrt{3}\times\sqrt{3}}$$

$$= \frac{2\sqrt{3}}{\sqrt{9}}$$

$$= \frac{2\sqrt{3}}{3}$$

b) $\frac{3}{\sqrt{5}}$ multiply both numerator and denominator by $\sqrt{5}$.

$$\frac{3}{\sqrt{5}} \times \frac{\sqrt{5}}{\sqrt{5}}$$

$$= \frac{3\sqrt{5}}{\sqrt{5}\times\sqrt{5}}$$

$$= \frac{3\sqrt{5}}{\sqrt{25}}$$

$$= \frac{3\sqrt{5}}{5}$$

Now that we have gone through a few examples, attempt the questions on the following activity.

Activity (*1.6*) Surds

Questions

Simplify the following

a) $5\sqrt{12} + 2\sqrt{3}$

b) $4\sqrt{2} - \sqrt{18}$

c) $\sqrt{3} \times \sqrt{5}$

d) $\frac{4}{\sqrt{3}}$

e) $\sqrt{3} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{27}}$

f) $5\sqrt{2} - 3\sqrt{8}$

g) $3\sqrt{28} - \sqrt{7}$

h) $(2+\sqrt{3})(4+\sqrt{3})$

i) $(2+\sqrt{3})(2-\sqrt{3})$

j) $\frac{4+\sqrt{3}}{2-\sqrt{3}}$ (***Hint:*** multiply both numerator and denominator by ($2+\sqrt{3}$))

Answers

a) $12\sqrt{3}$

b) $\sqrt{2}$

c) $\sqrt{15}$

d) $\frac{4\sqrt{3}}{3}$

e) $\frac{11\sqrt{3}}{9}$

f) $-\sqrt{2}$

g) $5\sqrt{2}$

h) $11+6\sqrt{3}$

i) 1

j) $11+6\sqrt{3}$

1.5 APPROXIMATIONS, ESTIMATIONS AND LIMITS OF ACCURACY

When shopping groceries, you sometimes estimate the amount you are going to pay at the till. An estimate gives a rough idea of how much you are going to pay. Approximations and estimations are usually done using place values, decimal places and significant figures

Place values

Given a number like 4975,683 from your Level 1 you established that every digit in this number has a place value for example 6 is a tenth. Let us look at rounding of numbers using place values.

Worked Example [18]

Questions

Round off 74,685 to the nearest

a) Tenth

b) Hundredth

c) Ten

d) Whole number

Solutions

a) 74,7 note that the tenth value we are interested in is 6, we look at the number 6, if it's below 5 we round down and when it's 5 and above we round up.

b) 74.69

c) 70

d) 75

Decimal Places

Hope you are familiar with decimal places from your level 1. Decimal places are the number of digits after the comma. For instance 50,697 has 3 decimal places.

Worked Example [19]

Questions

Round off the number 5,545 to

a) 1 decimal place b) 2 decimal places c) 3 decimal places

Solutions

a) 5,5 b) 5,55 c) 5,545

Significant figures (s.f)

Numbers can also be rounded off using significant figures. Significant figures are counted from the first non-zero digit. The number 0, 040 7 has 3 significant figures, that is, 4,0 and7, the first 2 zeros are place holders.

Worked Example [20]

Questions

1) Round off the number 0,008 046 to

   a) 1 s.f b) 2 s.f c) 3 s.f

2) Round off the number 475,063 to

   a) 1 s.f b) 2.s.f c) 3 s.f

d) 4 s.f e) 5 s.f

Solutions

1) a) 0,008 b) 0,008 0 c) 0,008 05

2) a) 500 b) 480 c) 475 d) 475,1 e) 475,06

Estimations

Any answer for a given question can be given as an approximate solution and it is called an estimate.

Worked Example [21]

Questions

1) Estimate each number to 1s.f and evaluate the following

a) 23×34 b) $0{,}8 \times 4{,}75$

c) $70{,}45 + 51{,}05$ d) $168{,}3 + 42{,}08$

e) $123 - 77$ f) $346{,}40 - 298{,}89$

2) By first rounding off each number to the nearest whole number, estimate the value of $\frac{7{,}03\times19{,}8-8{,}47}{10{,}84}$

Solutions

1) a) $23 \times 34 = 20 \times 30 = 600$ b) $0{,}8 \times 4{,}75 = 0.8 \times 5 = 4$

c) $70{,}45 + 51{,}05 = 70 + 50 = 120$ d) $168{,}3 + 42{,}08 = 200 + 40 = 240$

e) $123 - 77 = 120 - 80 = 40$ f) $346{,}40 - 298{,}89 = 300 - 300 = 0$

2) $\frac{7{,}03\times19{,}8}{10{,}84}$

$=\frac{7\times20-8}{11}$ Rounding off each number to the nearest whole number

$=\frac{140-8}{11}$

$=\frac{132}{11}$

$= 12$

Limits of accuracy

In most cases we tend to give our ages as whole numbers. We rarely give our ages as 24 years 2 months 3 weeks but we just say 24 years old.

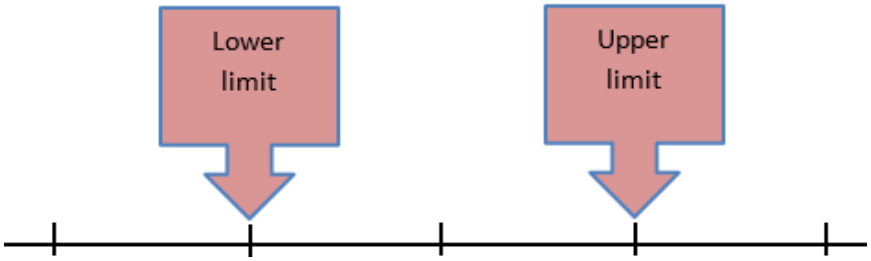
Worked Example [22]

Question

Chipo is asked her age and she gives her age as 17 years to the nearest whole number. What are the limits of her age?

Solution

Limits give us the smallest and the greatest number that can be rounded off to 17 years. The lower limit is 16.5 and the upper limit is greatest value less than 17.5 as illustrated on the diagram below.



The answer is given as an inequality

$16.5 \leq A < 17.5$ where A is the approximated age

Worked Example [23]

Questions

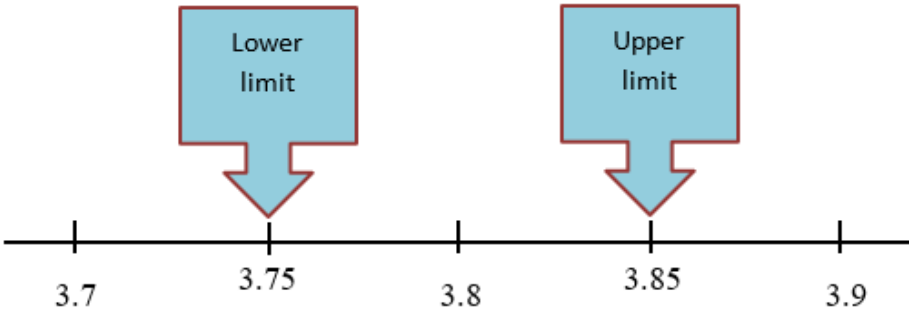
The sides of a square are given as 3.8cm to 1 decimal place.

a) Write down the limits of the length of the sides of the square in the form $a \leq l < b$

b) Hence calculate the least possible perimeter of the square.

Solutions

1) a) The lower limit is 3.75 and the upper limit is 3.85 as illustrated on the diagram below



$3.75 \leq l < 3.85$

b) Least possible perimeter 3.75×4 (use the lower limit)

=15 cm

Now that we have gone through some examples, attempt the questions on the following activity.

Activity (*1.1*) Rational and Irrational numbers

Questions

1) Express 0,040 56 to

a) 3 decimal places b) 2 significant figures

2) Chipo gave her mass as 64kg to the nearest kg. Write down the limits of her mass in the form $a \leq m < b$

3) By rounding each of the numbers to 1 significant figure, estimate the value $of\ \frac{384 \div 846}{\sqrt{(675-624)}}$

Answers

1) (a) 0.041 (b) 0.041

2) 63.5≤m<64.5

3) 0,05

1.6 ORDINARY AND STANDARD FORM

In life we meet very small or very large numbers, for instance the size of a virus or the distance between the sun and the earth. Such numbers are best understood if they are written in standard form. Standard form is also known as standard index form or sometimes as scientific notation. It involves writing large numbers or very small numbers in terms of powers of 10.

Large Numbers

Generally when a number is in standard form it should be written in the form $A \times 10^n$ where A is a number between 1 and 10 thus $1 \leq A < 10$.

The number 2700 can be written in different ways:

$\cancel{27 \times 10^2}$ $\quad 2.7 \times 10^3 \quad$ $\cancel{270 \times 10^1}$ $\quad$ $\cancel{2700 \times 10^0}$ $\quad$ etc

However, only 2.7×10^3 is the only one in standard form since it agrees with the above condition that $1 \leq A < 10$

Worked Example [24]

Question

Represent the following ordinary numbers in standard form

a) 9 000 b) 43 000 c) 47.689 d) 84 670

Solutions

a) $9\,000 = 9 \times 1\,000 = 9 \times 10^3$

b) $43\,000 = 4{,}3 \times 10^4$

c) $47.689 = 4.768\,9 \times 10^1$

d) $84\,670 = 8.467 \times 10^4$

Ordinary Form

You should be able to change a number in standard back to its ordinary from.

Worked Example [25]

Questions

Express the following numbers in ordinary form

1) 4.37×10^3
2) 8×10^{-3}
3) 4.37×10^{-3}
4) 4.365×10^{-1}
5) 4.37×10^{-9}

Solutions

1) 4.37×10^3

$= 4.37 \times 1000$

$= 4\ 370$

2) 8×10^{-3}

$= 8 \times \frac{1}{10^3}$

$= \frac{8}{10^3}$

$= \frac{8}{1000}$

$= 0.008$

3) 4.37×10^{-3}

$= 4.37 \times \frac{1}{10^3}$

$= \frac{4.37}{1000}$

$= 0.00437$

4) 4.365×10^{-1}

$= 4.365 \times \frac{1}{10^1}$

$= \frac{4.365}{10}$

$= 0.4365$

5) 4.37×10^{-9}

$= 4.37 \times \frac{1}{10^9}$

$= \frac{4,37}{1\ 000\ 000\ 000}$

= 0, 00000000437

Addition and Subtraction

Numbers in standard form can be added or subtracted. There are two methods of adding and subtracting numbers in standard form:

i. Changing to ordinary

ii. Factorisation

Worked Example [26]

Questions

Simplify the following leaving your answer in standard form.

1) $3.8 \times 10^3 + 4.17 \times 10^4$

2) $4.37 \times 10^4 - 2.3 \times 10^3$

3) $4.8 \times 10^{-3} - 2.3 \times 10^{-3}$

Solution

1) $3.8 \times 10^3 + 4,17 \times 10^4$

<u>Method 1:</u> By changing to ordinary form

$= 3.8 \times 1000 + 4.17 \times 10\ 000$

$= 3\ 800 + 41\ 700$

$= 45\ 500$

$= 4.55 \times 10^4$

Method 2: Factorisation

$3.8 \times 10^3 + 4.17 \times 10^4$

Identify a common factor and write it outside the brackets:

$= 10^3(3.8 + 4.17 \times 10)$

$= 10^3(3.8 + 41.7)$

$= 10^3(45.5)$

$= 45.5 \times 10^3$

45.5 is not between 1 and 10 so change it to standard form.

$= 4.55 \times 10^1 \times 10^3$

$= 4.55 \times 10^4$

2) $4.37 \times 10^4 - 2.3 \times 10^3$

Method 1: By changing to ordinary form

$= 4.37 \times 10\,000 - 2.3 \times 1\,000$

$= 43\,700 - 2\,300$

$= 41\,400$

$= 4.14 \times 10^4$

Method 2: Factorisation

$4.37 \times 10^4 - 2.3 \times 10^3$

$=10^3(4.37 \times 10 - 2.3)$

$= 10^3(43.7 - 2.3)$

$= 10^3\,(41.4)$

$= 41.4 \times 10^3$

$= 4.14 \times 10^1 \times 10^3$

$= 4.14 \times 10^4$

3) $4{,}8 \times 10^{-3} - 2{,}3 \times 10^{-3}$

<u>Method 1</u> Changing to ordinary form

$= 0.0048 - 0.0023$

$= 0.0025$

$= 2.5 \times 10^{-3}$

<u>Method 2</u> Factorisation

$4.8 \times 10^{-3} - 2.3 \times 10^{-3}$

$= 10^{-3}(4.8 - 2.3)$

$= 10^{-3}(2.5)$

$= 2.5 \times 10^{-3}$

Multiplication and division

When multiplying and dividing numbers in standard form multiply and divide like terms for instance

$a \times 10^{x} \times b \times 10^{n}$

$= a \times b \times 10^{x} \times 10^{n}$

Worked Example [27]

Questions

Simplify the following

1) $6 \times 10^{4} \times 7 \times 10^{5}$

2) $\frac{4{,}2 \times 10^{-3}}{2{,}1 \times 10^{-4}}$

Solutions

1) $6 \times 10^{4} \times 7 \times 10^{5}$

$= 42 \times 10^{4+5}$

$= 4.2 \times 10^{1} \times 10^{9}$

$= 4.2 \times 10^{10}$

2) $\frac{4.2 \times 10^{-3}}{1.2 \times 10^{-4}}$

$= \frac{4.2}{2.1} \times \frac{10^{-3}}{10^{-4}}$

$= 2 \times 10^{-3-(-4)}$

$= 2 \times 10^{1}$

Now that we have gone through some examples, attempt the questions on the following activity.

Activity (*1.8*) Ordinary and standard form

Questions

1) Express the following numbers in standard form
   a) 472
   b) 0.472
   c) 0.0057
   d) 0.00000483
   e) 38.043

2) Write the following in ordinary form
   a) 5.31×10^3
   b) 5.31×10^{-3}

3) Given that a =4.2× $10^3$and b =2.1× 10^2. Find
   a) a+b
   b) a –b
   c) ab
   d) $\frac{a}{b}$
   e) a^2

Answers

1) (a)4.72× 10^2

(b)4.72× 10^{-1}

(c) 5.7× 10^{-3}

(d)4.83×10^{-6}

(e) 3.8043× 10^1

2) (a)5310

(b) 0.00531

3) (a) 4.41×10^3

(b) 3.99×10^3

(c) 8.82×10^5

(d) 2×10^1

(e) 1.764×10^7

1.7 NUMBER BASES

Given that a number like 4784 can be written as

$$4000+700+80+4$$

$$=4 \times 10^3 + 7 \times 10^2+8 \times 10^1+4\times 10^0$$

4784 has been given in expanded form as a base of ten so 4784 can be written as

$$4784_{\text{ten}} \text{ or } 4784_{10}$$

Ordinary numbers that we deal with everyday are in base ten (denary base). Numbers can be converted from one base to another, that is, they can be converted from base 10 to bases 2, 3, 4, 5, 6, 7, 8 or 9 and vise versa.

Expanding numbers in a given base

Let us look into expanding numbers in a given base

Worked Example [28]

Questions

Expand the following

a) 5632_{ten} b) 10111_{two} c) 43211_5

Solutions

a) $5632_{\text{ten}}=5\times 10^3+6\times 10^2+3\times 10^1+2\times 10^0$

b) $10111_{\text{two}}=1\times 2^4+0\times 2^3+1\times 2^2+1\times 2^1+1\times 2^0$

c) $43201_{\text{five}}=4\times 5^4+3\times 5^3+2\times\ 5^2+0\times 5^1+1\times 5^0$

Converting numbers in base ten to any other base.

You should be able to convert a number in base ten to any other base.

Worked Example [29]

Questions

Convert 47_{ten} to base

a) 2 b) 5 ***c)*** 8

Solutions

a) 47_{ten} to base two

divide 47 by 2 and take the remainders from bottom upwards

	47		
2	23	r	1
2	11	r	1
2	5	r	1
2	2	r	1
2	1	r	0
2	0	r	1

$=101111_{\text{two}}$

b) 47_{ten} to base five

	47		
5	9	r	2
5	1	r	4
5	0	r	1

$= 142_{\text{five}}$

c) 47_{ten} to base eight

	47		
8	5 r	r	7
	7		
8	0 r	r	5
	5		

$= 57_{\text{eight}}$

Converting from any base to base ten

When converting a number to Base ten,you expand the number and simplify.

Worked Example [30]

Questions

Expand and convert the following to Base ten

a) 142_{five}
b) 3201_{four}
c) 101111_{two}

Solutions

a) 142_{five}

$=1\times 5^2+4\times5^1+2\times5^0$

$=25+20+2$

$= 47_{\text{ten}}$

b) 3201_{four}

$=3\times4^3+2\times4^2+0\times4^1+1\times4^0$

$=192+32+0+1$

$=225_{\text{ten}}$

c) 101111_{two}

$=1\times2^5+0\times2^4+1\times2^3+1\times2^2+1\times2^1+1\times2^0$

$=32+0+8+4+2+1$

$=47_{\text{ten}}$

Converting from any base to another base

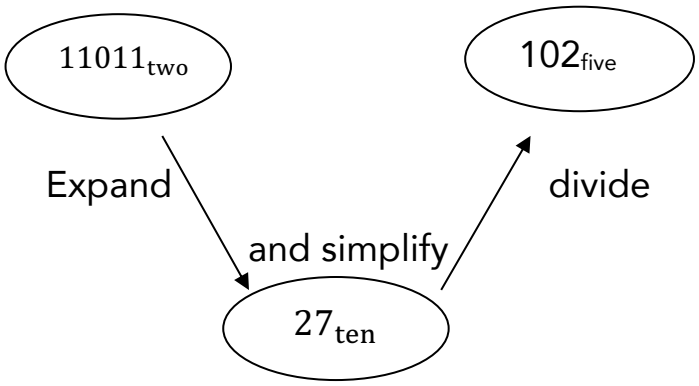
When changing a number that is not in base ten for instance from base two to base five, you need to pass through base ten.

Worked Example [31]

Questions

Convert 101111_{two} to base five

Solutions



11011_{two}

$=1\times2^4+1\times2^3+0\times2^2+1\times2^1+1\times2^0$

$=16+8+0+2+1$

$= 27_{ten}$

Then convert 27_{ten} to base 5

5	27	
5	5 r	2
5	1 r	0
	0 r	
1		

$= 102_{five}$

Addition and Subtraction

You have observed from previous examples that when a number is in base two, it only has ones and zeros, that is, it has digits less than 2. Moreso, a number in base 5 the digits should be less than 5.

Worked Example [31]

Question

Evaluate $434_{\text{five}} + 423_{\text{five}}$ giving your answer in base five.

Solution

Method 1

$434_{\text{five}} + 423_{\text{five}}$

$$\begin{array}{r} 434_5 \\ +423_5 \\ \hline 1412_5 \\ \hline \end{array}$$

Stage1: 4+3=7 , divide $\frac{7}{5}$= 1 r 2 put down the remainder and carry 1

Stage2: 3+2+1=6 , divide $\frac{6}{5}$ = 1 r 1 put down the remainder and carry 1

Stage3: 4+4+1=9, divide $\frac{9}{5}$=1 r 4 put down 4 and carry 1

Method 2

$434_{\text{five}}+423_{\text{five}}$

Converting to base ten

434_{five} to base 10

$= 4\times5^2+3\times5^1+4\times5^0$

$= 100+15+4$

$= 119_{\text{ten}}$

Converting to base ten

423_{five} to base 10

$= 4 \times 5^2 + 2 \times 5^1 + 3 \times 5^0$

$= 100 + 10 + 3$

$= 113_{\text{ten}}$

$434_{\text{five}}+423_{\text{five}} = 119_{\text{ten}} + 113_{\text{ten}} = 232_{\text{ten}}$

Then convert 232_{ten} to base 5

5	232	
5	46	r
	2	
5	9	r
	1	
5	1	r
	4	
	0	

r 1

$= 1412_{\text{five}}$

NB Method 2 is suitable when adding and subtracting numbers that have different bases

Worked Example [33]

Question

$4312_{\text{five}} - 434_{\text{five}}$

Solution

$4312_{\text{five}} - 434_{\text{five}}$

$$\begin{array}{r} 4312_5 \\ -\ 434_5 \\ \hline 3323_5 \\ \hline \end{array}$$

Stage 1: 2 – 4 is not possible, so add 5 to 2 to get 7, then 7 – 4 = 3, write 3

Stage 2: add 1 to 3 to get 4, then 1 – 4 is not possible, then add 5 to 1, then 6 – 4 = 2, write 2

Stage 3: add 1 to 4 to get 5, then 3 –5 is not possible, then add 5 to 3, then 8 – 5 = 3, write 3

Stage 4: add 1 to 0 to get 1, then 4 – 1 = 3, then write 3

Alternatively, you can subtract by first changing the numbers to base ten. Try the process until you get the answer 3323_{five}.

Now that we have gone through a few examples, attempt questions in the following activity.

Activity (*1.9*) Number bases

Questions

1) Convert 97_{ten} to
   a) base five
   b) base eight

2) Convert each of the numbers to base ten
   (a) 4112_{five}
   (b) 413_{eight}

3) Convert
   a) 213_{five} to base two
   b) 416_{eight} to base five

4) Simplify
   a) $413_{\text{five}}+143_{\text{five}}$
   b) $1100_{\text{two}} - 111_{\text{two}}$
   c) $431_{\text{five}}+ 450_{\text{six}}$ [***Hint:*** Give answer in base ten]

Answers

1) a) 342_{five} b) 141_{eight}
2) a) 532_{ten} b) 267_{ten}
3) a)111010_{two} b) 2040_{five}
4) a)1111_{five} b) 101_{two} c) 290_{ten}

1.8 SCALE AND SIMPLE MAP PROBLEMS.

Scale on maps and plans is given as a ratio in the form 1:n . For example 1:500 means 1cm represents 500cm or 1cm represents 5m.

If on a map the scale is given as 1:50 000 that means 1cm represents 50 000cm therefore 1cm represents 500m, then 1cm represents 0.5km.

The scale for length can be changed to become scale for area by squaring both sides. For instance if the scale for length is 2cm represents 5m then the scale for area becomes $4cm^2$ represents $25m^2$

Worked Example [34]

Questions

The scale on a building plan is given as 1:500

a) Find the actual length of a window on the map that has length 8cm in metres.

b) The actual length of the building is 30m. Find the length on the map.

c) Area of one of the rooms of a house on the plan is $18cm^2$. Find the actual area of the room in m^2.

Solutions

a) 1:500 means 1cm represent 500cm then 1cm represents 5m

1cm represents 5m

8cm represents ?

$\frac{8cm}{1cm} \times \frac{5m}{1} = 40m$

b) 5m represents 1cm

30m represents ?

$\frac{30m}{5m} \times 1cm = \underline{6cm}$

c) Scale for area

$= (1cm)^2$ represents $(5m)^2$

$=1cm^2$ represents $25m^2$

$=18cm^2$ represents ?

$=\frac{18cm^2}{1cm^2} \times 25m^2 = 450m^2$

Activity (*1.10*) Scale

Questions

The scale on a map is given as 1cm represents 5km

a) Write the scale in the form 1:n.

b) If the distance between two towns Rusape and Mutare is 16cm on the map, find the actual distance between Rusape and Mutare in km.

c) The area of an airport on a map is $12cm^2$, calculate the actual area of the airport in km^2.

Answers

(a) 500000 (b) 80 km (c) $300km^2$

1.9 NUMBER PATTERNS AND SEQUENCES

A sequence is a set of numbers which follows a well defined order or rule,for example

1;2 ;4;8;16;$\cdots$ or 2;4;6;8;….

Worked Example [35]

Questions

a) Write the next 3terms of the sequence 1;3;5;7 . . .

b) Write the next 3 terms of the sequence 2;4;8;16 . . .

Solutions

a) 9;11;13 In this case the sequence is a set of odd numbers being obtained by adding 2.

b) 32;64;128 we obtain the next term by multiplying the previous by 2

1.10 Summary

From this unit you have realised that different types of numbers can be added, subtracted, multiplied and divided differently. This unit is acting as the basis of understanding the next units. You can also use the skills you have acquired in this unit in other learning areas. If you have mastered the skills in this unit then move to the next unit.

1.11 Further Reading

1. Macrae, M, Madungwe, L and Mutangadura, A (2017) New General Mathematics Book 3, Pearson Capetown.
2. Macral, M, Madungwe, L and Mutangadura, A (2017) New General Mathematics Book 4, Pearson Capetown.
3. Meyers, C; Graham, B; Dawe, L (2004) Mathscape Working Mathematically, 9th Ed, MacMillan
4. Pimentel R and Wall T (2011) International Mathematics, Hodder UK
5. Rayner D (2005) Extended Mathematics. Oxford New York

1.12 Assessment Test

Answer the following questions,do not use a calculator when answering these questions

1) Express 43 786
   a) in standard form (1)
   b) to the nearest thousand (1)
   c) to three significant figures (1)

2) Write 0,085
   a) in standard form (1)
   b) as a fraction in its lowest terms (1)
   c) to two decimal places (1)

3) Evaluate

a) $1\frac{3}{4} - \frac{2}{5} + 5\frac{1}{3}$ (1)

b) 0.06×0.003 (1)

c) $\frac{(0,3)^2 \times 0,2}{0,06}$ (1)

4) a) Write 3,65hours in hours and minutes (1)

b) The price of a radio is $270. This price is 20% more than last year's price. Calculate last year's price of the radio (2)

5) a) Write down the largest 4digit number in base 9 (1)

b) Convert 412_{ten} to base eight (1)

c) Convert 112_{five} to base three (1)

6) Given that $a = 9 \times 10^8$, $b = 6 \times 10^5$. Find the value of

a) a+b (1)

b) ab (1) $\frac{a}{b}$ (1)

c) $\sqrt{a}$ (1)

7) The diameter of a circle is given as 9,4cm to 1 decimal place

a) (a)write down the limits of the diameter in the form $a \leq d < b$ (2)

b) (b)calculate the least possible circumference of the circle in terms of π (2)

8) a) Of the livestock on the farm, 70% are cattle and the rest are goats. If the farmer has 18 goats, calculate the total livestock on the farm (2)

b) When making a concrete mixture of cement,sand and concrete the ratio 2:3:4 is used respectively. Calculate the number of wheelbarrows of sand required if there are 6 bags of cement (2)

9) Tendai and Tanaka are 12years and 30years respectively

a) Write down their ages as a ratio in its simplest form in the form a:b (1)

b) If they share some money in the ratio of their ages ,and Tendai got $24. How much did Tanaka get? (2)

10) The scale on a map is given as 1:250

a) Calculate the actual length of a church building in metres if it is 8cm on the map (2)

b) The area of a room on the map is 12cm^2. Calculate the actual area of the room in m^2 (2)

Answers

1)	(a) 4.3786×10^4	(b) 44000	(c) 43800
2)	(a) 8.5×10^{-2}	(b) $\frac{17}{200}$	(c) 0.09
3)	(a) $6\frac{41}{60}$	(b) 0.00018	(c) 0.3
4)	(a) 3hrs 39 minutes	(b) $225	
5)	(a) 8888	(b) 634_{eight}	(c) 1012_{three}
6)	(a) 9.006×10^8	(b) 5.4×10^{14}	c)1.5×10^3 (d)3×10^4
7)	(a) $9.35\leq d<9.45$	(b) 9.35π	
8)	(a)60	(b)9	
9)	(a)2:5	(b)$60	
10)	(a)20m	(b)$75\mathrm{m}^2$	

UNIT 2 - ALGEBRA 1

CONTENTS

2.1 INTRODUCTION

This unit introduces the topic on Algebra which is part of mathematics in which letters and other general symbols are used to represent numbers and quantities in formulae and equations. This unit seeks to equip you with the skills required in dealing with manipulation of algebraic expressions. Algebraic manipulation is one of the most basic, necessary and important skills which should be in a problem solver's stock, as without it a problem solver would hopelessly be stuck on innumerable problems. The skill of algebraic manipulation is acquired through practice and solving problems.

OBJECTIVES

After going through this unit you should able to :

- manipulate algebraic processes
- find the lowest common multiple (L.C.M) and highest common factor (H.C.F) of algebraic expressions
- simplify algebraic fractions

KEY TERMS

Algebra:– use of letters of the alphabet to stand for numbers

Expand:– remove brackets from an expression by multiplication

Algebraic expression: – a mathematical statement were letters and numbers are used together for example $2x$; $4a + 5$; $0{,}5y + 6z$

Factorise:– writing an expression in terms of its factors

Substitute:– to replace a letter with a number

Quadractic Expression: – an expression where highest power of the unknown is 2.

TIME: should not spend not more than 10 hours in this unit.

STUDY SKILLS

In order to understand activities in this unit you should be able to deal with directed numbers. The skill of understanding algebraic manipulation is acquired through practice and solving problems. You should attempt to solve as many problems as possible.

2.2 LOWEST COMMON MULTIPLE (L.C.M) AND HIGHEST COMMON FACTOR (H.C.F)

2.2.1 Lowest common multiple

Lowest common multiple is the smallest number that can be divided by the given numbers without leaving a remainder.

Given the numbers 12 and 18,you can find the lowest common multiple by first listing the multiples of 12 and 18 and identifying the smallest whole number which is divisible by both numbers.

Multiples of $12 = 12;\ 24;\ \mathbf{36};\ 48\ \ldots$

Multiples of $18 = 18;\ \mathbf{36};\ 54;\ 72\ \ldots$

$\mathrm{L.C.M} = 36$

You can also find the LCM through the use of prime factors. To find the LCM of 12 and 18 you start by writing each number as a product of its prime factors.

$12 = 2 \times 2 \times 3$

$18 = 2 \times 3 \times 3$ Take every factor where it is appearing the highest number of times

$\mathrm{L.C.M} = 2 \times 2 \times 3 \times 3$

$= 36$

The same method can be used when dealing with algebraic expressions. For example you may be asked to find the L.C.M of $12a^2b$ and $18ab^2$.
This is done by splitting the two into their respective prime factors as shown below:

$12a^2b = 2 \times 2 \times 3 \times a \times a \times b$

$18ab^2 = 2 \times 3 \times 3 \times a \times b \times b$

$L.C.M = 2 \times 2 \times 3 \times 3 \times a \times a \times b \times b$

$= 36a^2b^2$

2.2.2 Highest common factor (H.C.F)

The highest common factor (HCF) of two whole numbers is the largest whole number which is a factor of both. To find H.C.F of 12 and 18 you list the factors of 12 and 18 as shown below:

Factors of $12 = 1; 2; 3; 4; 6; 12$

Factors of $18 = 1; 2; 3; 6; 9; 18$

Common factors (C.F)= $1; 2; 3; 6$

$H.C.F = 6$

Alternatively H.C.F can be found by using prime factors as shown below:

$12 = 2 \times 2 \times 3$

$18 = 2 \times 3 \times 3$

$H.C.F = 2 \times 3$

$H.C.F = 6$

Worked Example [1]

Question

Find H.C.F of $18ab^2$ and $24a^2b$

Solution

$18ab^2 = 2 \times 3 \times 3 \times a \times b \times b$

$24a^2b = 2 \times 2 \times 2 \times 3 \times a \times a \times b$

$C.F \quad = 2 \times 3 \times a \times b$
$H.C.F = 6ab$

2.3 ALGEBRAIC MANIPULATIONS

Have you ever encountered an algebraic expression? If yes list any two algebraic expressions in the space provided below

Here are some examples of algebraic expressions, $3x + y, 4z - 5e$ and $5p - 3r + 4w$

2.3.1 Linear expressions

Linear expressions are algebraic expressions that contain a variable, and the highest power of the unknown is 1. For example in the expression $2x + 4, x$ is the variable and the highest power of x is 1.When simplifying linear expressions we collect like terms, see the following example

Worked Example [2]

Question

Simplify the following expression, $3x + 4y - 8x + 7y + 9$

Solution

$3x + 4y - 8x + 7y + 9$
$= 3x - 8x + 4y + 7y + 9$ collecting like terms
$= -5x + 11y + 9$

2.3.1.1 Expressions with brackets

When removing brackets, the term outside the brackets multiplies each term inside the brackets.

- ✓ $a(b + c) = ab + ac$
- ✓ $-a(b + c) = -ab - ac$ [**Hint:** If the term outside the brackets has a negative sign the signs inside the brackets change.]
- ✓ $(a + b)(m + n) = am + an + bm + bn$

Now, let us look at the example below on how to simplify expressions with brackets.

Worked Example [3]

Questions

Simplify the following expressions

(a) $2x(3y - 4x + 5)$

(b) $3x(2 + m) - 4(3 - y)$

(c) $6 - 4(x + y)$

(d) $(3x + y)(2a + b)$

(e) $-2x^2(x + 3y - \frac{1}{x})$

(f) $\frac{-2}{x}(-x + 4y + \frac{1}{x}$

Solutions

(a) $2x(3y - 4x + 5)$

$= 6xy - 8x^2 + 10x$ [no like terms so leave the answer like that]

(b) $3x(2 + m) - 4(3 - y)$

$= 6x + 3mx - 12 + 4y$ [no like terms]

(c) $6 - 4(x + y)$

$= 6 - 4x - 4y$ [Remember to multiply before subtracting]

(d) $(3x + y)(2a + b)$

$3x(2a + b) + y(2a + b)$

$6ax + 3bx + 2ay + by$

(e) $-2x^2(x + 3y - \frac{1}{x})$

$-2x^3 - 6x^2y + 2x$

(f) $\frac{-2}{x}(-x + 4y + \frac{1}{x}$

$$2 - \frac{8y}{x} - \frac{2}{x^2}$$

2.3.1.2 Expressions with fractions

Just like the ordinary fractions you dealt with in Unit 1, expressions with fractions can be added or subtracted.

Worked Example [4]

Questions

Simplify the following fractions

a) $\frac{3}{a} + \frac{4}{b} - \frac{5}{c}$

b) $\frac{2x+3}{5} + \frac{x+1}{4}$

Solutions

a) $\frac{3}{a} + \frac{4}{b} - \frac{5}{c}$ [common denominator abc]

$= \frac{3bc + 4ac + 5ab}{abc}$ [put under the common denominator]

b) $\frac{2x+3}{5} + \frac{x+1}{4}$ [common denominator 20]

$= \frac{4(2x+3) + 5(x+1)}{20}$

$= \frac{8x+12+5x+5}{20}$

$= \frac{13x+17}{20}$

Now that we have gone through the examples above, attempt the following activity.

Activity (*2.1*) Algebraic Expressions

Questions

1. Simplify the following
   a) $a - 3b + 4c - 10a + 5b$
   b) $4(m + 3n - p)$
   c) $2(x + 3y) + 4(x - 5y)$
   d) $2a(a + 3) + 3(2a + 4)$
   e) $(3x - y)(2x - 3y)$

2. Write the followung as single fractions
   a) $\frac{2a+3}{4} + \frac{a-1}{4}$
   b) $\frac{x-2}{3} - \frac{2x+3}{5}$
   c) $\frac{m-3}{4b} - \frac{m-2}{8b} + 5$

Answers

1 (a) $-9a + 2b + 4$ (b) $4m + 12n - 4p$

(c) $6x - 14y$ (d) $2a^2 + 12a + 12$

(e) $6x^2 - 11xy + 3y^2$

2 (a) $\frac{3a+2}{4}$ (b) $\frac{-x-19}{15}$ (c) $\frac{m-4+40b}{8b}$

TIP: common denominator should be lowest commom multiple of the denominators.

2.3.2 Factorisation

An expression can be written in terms of its factors. This process is called factorisation.

To factorise the expression 6b + 9ab, you write the H.C.F outside the bracket and divide each term inside the bracket by the H.C.F.

6b + 9ab

$= 3b\left(\frac{6b}{3b} + \frac{9ab}{3b}\right)$

$= 3b(2 + 3a)$

Whenever you are factorising, first identify the H.C.F that you are going to write outside the brackets and then divide each term by the H.C.F

2.3.2.1 Factorising two terms

When factorising two terms, identify the H.C.F and write it outside the brackets as shown below.

Worked Example [5]

Question

Factorise 8ab+12a

Solution

8ab+12a

$=4a\left(\frac{8ab}{4a} + \frac{12a}{4a}\right)$

$=4a(2b+3)$

2.3.2.2 Factorising 4 terms

When factorising four terms group the terms in pairs and then factorise as shown below.

Worked Example [6]

Questions

Factorise

(a) $ax + ay + bx + by$

(b) $6a^2 - 3a + 4a - 2$

(c) $6mn + bd - 3bm - 2nd$

Solutions

(a) $ax + ay + bx + by$

$= a(x + y) + b(x + y)$

[(x+y) is now common]

$= (x + y)\,(a + b)$

(b) $6a^2 - 3a + 4a - 2$

$= 3a(2a - 1) + 2(2a - 1)$

$= (2a - 1)(3a + 2)$

(c) $6mn + bd - 3bm - 2nd$

[Regroup because the first two terms have no common factor]

$= 6mn - 3bm - 2nd + bd$

$= 3m(2n - b) - d(2n - b)$

$= (2n - b)(3m - d)$

2.3.2.3 Factorising 3 terms (Quadratic expressions)

Using the Product, Sum and Factor method

Expressions of the form $ax^2 + bx + c$ (quadratic expressions) are factorised using Product, Sum and Factor method (P.S.F), as shown in the example below.

Worked Example [7]

Questions

Factorise the following expressions

a) $x^2 + 7x + 12$ b) $3x^2 - 4x + 1$ c) $a^2 - a - 20$

Solutions

a) $x^2 + 7x + 12$

$P = x^2 \times 12 = 12x^2$

$S = 7x$

$F = 4x + 3x$

The factors (F) should give a product (P) and sum (S).

Replace $7x$ with $4x + 3x$ so that the terms become 4

$= x^2 + 4x + 3x + 12$

$= x(x + 4) + 3(x + 4)$

$= (x + 3)(x + 4)$

b) $3x^2 - 4x + 1$

$P = 3x^2 \times 1 = 3x^2$

$S = -4x$

$F = -3x - x$

$= 3x^2 - 3x - x + 1$

$= 3x(x - 1) - 1(x - 1)$

$= (3x - 1)(x - 1)$

c) $a^2 - a - 20$

$P = a^2 \times -20 = -20a^2$

$S = -a$

F=-5a+4

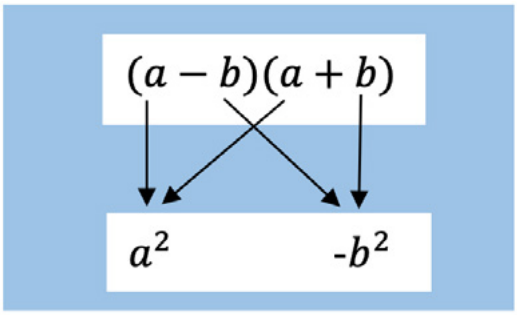
$= a^2 - 5a + 4a - 20$

$= a(a - 5) + 4(a - 5)$

$= (a + 4)(a - 5)$

Factorising difference of two squares.

When two square numbers are subtracting each other like $a^2 - b^2$, they can be factorised by splitting a^2 and $-b^2$ into their respective factors as shown below.



Worked Example [8]

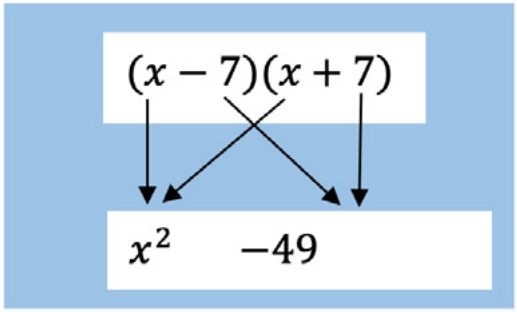
Questions

Factorise the following expressions

a) $x^2 - 49$ b) $2x^2 - 32$ c) $1 - 9x^2$

Solutions

a) $x^2 - 49 =$



b) $2x^2 - 32 = 2(x^2 - 16)$ [First pull out common factor 2]

$= 2(x - 4)(x + 4)$

c) $1 - 9x^2 = (1 - 3x)(1 + 3x)$

Now that we have gone through some example, you can now attempt questions on the following activity.

Activity (*2.2*) Factorisation

Questions

Factorise the following

1) $3m^2 - 12m$
2) $-10ab + 5a$
3) $x^2 + 6x + 8$
4) $3x^2 + 8x + 5$
5) $3m^2 - 48$
6) $a^2 + 15a + 50$
7) $3r^2 - 5r - 8$
8) $m^2 + 5mx - 3mx - 15x^2$
9) $81m - 3m^3$
10) $(a^2 - \frac{4}{9})$

Answers

1) $3m(m-4)$
2) $-5a(2b-1)$
3) $(x+4)(x+2)$
4) $(x+1)(3x+5)$
5) $3(m-4)(m+4)$
6) $(a+5)(a+10)$
7) $(r+1)(3r-8)$
8) $(m-3x)(m+5x)$
9) $3m(27 - m^2)$
10) $(a - \frac{2}{3})(a + \frac{2}{3})$

2.3.3 Substitution

Substitution refers to putting values where the letters are. It is also viewed as a process whereby variables are replaced by numerical values.

Worked Example [9]

Questions

Given that $a = 3, b = 2$ and $c = -1$ find the value of

a) $a + 3b$
b) abc
c) $\frac{a+b}{c}$
d) $\frac{a-c}{b+c}$
e) $(a - c)^b$

Solutions

a) $a + 3b = 3 + 3 \times 2$
$= 3 + 6$
$= 9$

b) $abc = 3 \times 2 \times -1$

$= -6$

c) $\frac{a+b}{c} = \frac{3+2}{-1} = \frac{5}{-1} = -5$

d) $\frac{a-c}{b+c} = \frac{3-(-1)}{2+(-1)}$

$= 4$

e) $(a - c)^b = (3 - (-1))^2$

$= (3 + 1)^2$

$= 4^2$

$= 16$

Now that we have gone through the above examples, attempt the following activity.

Activity (*2.3*) Substitution

Questions

1. Given that a=4 and b= –1.

   Find the value of

   a) a +3b b) $a^2 - b^2$ c) $(a - b\}^2$

2. Given the expression $3x^2 - 4x + 5$

   Simplify the expression when

   a) x = 0 b) x = 1 c) x = –1

   d) x = 2 e) x = –2

3. If $m = nq - \frac{q}{n}$

   a) Find the value of m when n = 14 and $q = \frac{2}{7}$

   b) Find q when m = 8 and n = 3

Solutions

1	(a) 1	(b) 15	(c) 25		
2	(a) 5	(b) 4	(c) 12	(d) 9	(e) 25
3	(a) $3\frac{48}{49}$	(b) 3			

2.4 ALGEBRAIC FRACTIONS

Algebraic fractions can be written in their lowest terms, added, subtracted, multiplied and even be divided.

2.4.1 Lowest terms

To reduce an algebraic fraction to its lowest terms, factorise the numerator and the denominator then cancel out common factors as shown below.

Worked Example [10]

Questions

Simplify the following

a) $\frac{a^2+ab}{b^2+ab}$. b) $\frac{x^2-5x+6}{x^2-3x+2}$. c) $\frac{a^2+2ab+b^2}{a^2-2ab+b^2}$

Solutions

a) $\frac{a^2+ab}{b^2+ab} = \frac{a(a+b)}{b(b+a)}$ [factorising]

$\frac{a\cancel{(a+b)}}{b\cancel{(a+b)}}$

$=\frac{a}{b}$

b) $\frac{x^2-5x+6}{x^2-3x\ 2} = \frac{\cancel{(x-3)}(x-2)}{\cancel{(x-2}\ )(x-1)}$

$=\frac{(x-3)}{(x-1)}$

c) $\frac{a^2+2ab+b^2}{a^2-2ab+b^2} = \frac{\cancel{(a+b)}(a+b)}{a\cancel{(a-b)}(a-b)}$

$= \frac{(a+b)}{a\,(a-b)}$

TIP: To be able to reduce algebraic fractions to lowest terms you should know factorisation well.

2.4.2 Multiplication and division

To multiply or divide algebraic expressions you should be able to reduce the fractions using common factors.

Remember multiplication and division of fractions.

1) $\frac{a}{b} \times \frac{m}{n} = \frac{am}{bn}$
2) $\frac{a}{b} \div \frac{m}{n} = \frac{a}{b} \times \frac{n}{m} = \frac{an}{bm}$

Worked Example [11]

Questions

Simply the following

(a) $\frac{a^2-9}{a^2+5a+6} \times \frac{a^2+2a}{a^2}$ (b) $\frac{a^2-b^2}{a^2-2ab+b^2} \div \frac{a^2+ab}{b^2-ab}$

Solutions

(a) $\frac{a^2-9}{a^2+5a+6} \times \frac{a^2+2a}{a^2}$

$= \frac{(a-3)\cancel{(a+3)}}{\cancel{(a+3)}\cancel{(a+2)}} \times \frac{a\cancel{(a+2)}}{a^2}$ [**Hint:** factorise the numerators and the denominators]

$= \frac{a(a-3)}{a^2} = \frac{a-3}{a}$

(b) $\frac{a^2-b^2}{a^2-2ab+b^2} \div \frac{a^2+ab}{b^2-ab}$

$\frac{(a-b)(a+b)}{(a-b)(a-b)} \div \frac{a(a+b)}{b(b-a)}$ [factorise the numerators and denominators]

$\frac{\cancel{(a-b)}\cancel{(a+b)}}{\cancel{(a-b)}(a-b)} \times \frac{b(b-a)}{\cancel{a(a+b)}}$ [change ÷ to ×, then invert the fraction]

$= \frac{b(b-a)}{a(a-b)}$

$= \frac{-b\cancel{(a-b)}}{a\cancel{(a-b)}}$ [for b-a factor out –1 to get –1 (a–b) in the numerator]

$= \frac{-b}{a}$

Activity (*2.4*) Algebraic fractions

Questions

Simplify the following

1) $\frac{m^2+mn}{m^2-n^2}$

2) $\frac{15+2a-a^2}{a^2-25}$

3) $\frac{d^2-3d-4}{d^2-4d} \div \frac{d^2-4d+4}{d^2-4}$

4) $\frac{x^2-y^2}{xy+x^2} \times \frac{2x^2}{xy-x^2}$

5) $\frac{5}{a+5}+\frac{3}{a+2}$

Solutions

1) $\frac{m}{m-n}$

2) $\frac{-a-3}{a+5}$

3) $\frac{(d+1)(d+2)}{d(d-2)}$

4) -2

5) $\frac{8a+25}{(a+5)(a+2)}$

2.4.3 Addition and subtraction

Algebraic fractions can be added or subtracted by putting the fractions under a common denominator, then simplify as shown in following examples

Worked Example [12]

Questions

Simplify the following

(a) $\frac{3}{2x+2y}+\frac{4}{3x+3y}$

(b) $\frac{m+4}{2m-8}-\frac{m+3}{12-3m}$

Solutions

a) $\frac{3}{2x+2y}+\frac{4}{3x+3y}$

$\frac{3}{2(x+y)}+\frac{4}{3(x+y)}$ [factorise the denominator]

$=\frac{3(3)+2(4)}{6(x+y)}$ [find L.C.M of $2(x+y)$ and $3(x+y)$]

[$6(x+y)$, then put under one denominator]

$=\frac{17}{6(x+y)}$ [simplify the numerator]

b) $\frac{m+4}{2m-8} - \frac{m+3}{12-3m}$

Find the $\mathrm{L.C.M}$ of $2m - 8$ and $12 - 3m$ by first factorising the 2 expressions

$2m - 8 = 2(m - 4)$

$12 - 3m = -3(m - 4)$

Therefore $\mathrm{L.C.M} = -6(m - 4)$

Put the fractions under a common denominator and then simplify the numerator as shown below

$$\frac{-3(m+4) - 2(m+3)}{-6\,(m-4)}$$

$$\frac{-3m - 12 - 2m - 6}{-6(m-4)}$$

$$\frac{5m + 18}{6(m-4)}$$

Activity (*2.5*) Addition and subtraction

Questions

Simplify the following

a) $6 - \frac{m+n}{c}$

b) $\frac{7}{x-2} - \frac{3}{x-3}$

c) $\frac{5}{2(a+b)} - \frac{4}{3(a+b)}$

d) $\frac{3}{a^2-2a-8} + \frac{5}{a^2-6a+8}$

e) $\frac{7}{x^2-16} + \frac{3}{(x+4)^2}$

Answers

a) $\frac{6c-m-n}{c}$

b) $\frac{4x-15}{(x-2)(x-3)}$

c) $\frac{7}{6(a+b)}$

d) $\frac{8a+4}{(a-2)(a-4)(a+2)}$

e) $\frac{10x+16}{(x-4)(x+4)^2}$

2.5 Summary

From this unit you have learnt how to simplify different types of algebraic expressions, that is factorisation, finding H.C.F and LCM and simplification of algebraic expressions. The skills you have mastered will help you understand what is in store for you in the next unit.

2.6 Further Reading

1. Macrae, M, Madungwe, L and Mutangadura, A (2017) New General Mathematics Book 3, Pearson Capetown.
2. Macral, M, Madungwe, L and Mutangadura, A (2017) New General Mathematics Book 4, Pearson Capetown.
3. Meyers, C; Graham, B; Dawe, L (2004) Mathscape Working Mathematically, 9th Ed, MacMillan
4. Pimentel R and Wall T (2011) International Mathematics, Hodder UK
5. Rayner D (2005) Extended Mathematics. Oxford New York

2.7 Assessment Test

1. Simplify the following expressions.

   a) $5x-(3x-2)$ (2)

   b) $-3(2x-3y+5)$ (2)

   c) $(2x+3)^2$ (2)

2. Write the following as single fractions

   a) $\frac{2x-1}{3}+\frac{5x+2}{4}$

   b) $\frac{3x+4}{3}-\frac{7x+1}{5}$

   c) $\frac{5}{x+2}-\frac{3}{x-3}$ (6)

3. Express the following fractions in the lowest terms

   a) $\frac{2x-2y}{2y-2x}$

   b) $\frac{a^2+5a+6}{a^2-9}$

   c) $\frac{x^2-4}{x^2-6x+8}$ (6)

4. Factorise completely

a) $125x^3 - 5x$ (2)

b) $2x^2 - 25x - 27$ (2)

c) $a^2 + am - an - mn$ (2)

5. Simplyfy the following

a) $\frac{m+4}{m^2+8m+15} \times \frac{m^2+2m-3}{m^2-16}$ (3)

b) $\frac{a^2-b^2}{a^2-2ab+b^2} \div \frac{a^2+ab}{b^2-ab}$ (3)

6. The sides of a rectangle are (3x-4) cm and (2x + 1) cm

a) Calculate the perimeter of the rectangle in terms of x in its simplest terms (3)

b) Calculate the area of the rectangle in terms of x in its simplest form (2)

c) If (3x–4)cm and (2x+1) cm are sides of a square pipe find the value of x (2)

Answers

1. (a) $2x + 2$

   (b) $-6x + 9y - 15$

   (c) $4x^2 + 12x + 9$

2. (a) $\frac{23x+2}{12}$

   (b) $\frac{17-6x}{15}$

   (c) $\frac{2x-21}{(x+2)(x-3)}$

3. (a) -1

   (b) $\frac{a+2}{a-3}$

   (c) $\frac{x+2}{x-4}$

4. (a) $5x(5x-1)(5x+1)$

   (b) $(x+1)(2x-27)$

   (c) $(a-n)(a+m)$

5. (a) $\frac{m-1}{(m-4)(m+5)}$

   (b) $\frac{-b}{a}$

6. (a) $16x - 6$

   (b) $6x^2 - 5x - 4$

   (c) 5

UNIT 3 - SETS

CONTENTS

3.1 INTRODUCTION

This unit is a continuation of the topic on sets you did in Level 1. In this unit you are going to learn about the Set Builder notation and how to use Venn diagrams to solve problems with at most three subsets.

OBJECTIVES

After going through this unit you should be able to:

- define and describe set-builder notation by listing the elements.
- use Venn diagrams to show sets.
- use Venn diagrams to solve problems involving at most three sets and the universal set.

KEY TERMS

Set builder notation: - It is a shorthand method of describing a set and it involves use of symbols to describe elements of a set as well as their properties.

Venn diagrams: - This is another way of presenting sets using diagrams, a rectangle to represent universal set and circles to represent subsets.

TIME

You are advised not to spend more than 10 hours on this unit.

STUDY SKILLS

This topic builds on the topic on Inequalities. This means that for you to easily understand the topic on Sets, you should have gone through Inequalities. The key skill to mastery of mathematical concepts is practice. You need to solve as many problems in Sets as possible for you to grasp all the concepts in this topic.

3.2 SET BUILDER NOTATION

A Set builder notation is a shorthand method for describing sets and it involves use of symbols to describe elements of a set as well as their properties.

Remember (Set symbols at level 1)

The following table shows the symbols, their meaning and use in the topic on Sets.

Table 3.1 – Set symbols

SYMBOLS	MEANING	USE
ξ OR ɯ	Universal set	ξ = { months of the year }
{ - - - - }	A set of	W = { months of the winter }
∈	An element of	June = ∈ W
n(W)	Number of elements in set W	n(W) = 4
⊂	Is a proper subset of	X ⊂ W
⊆	Improper subset	If all elements of X are elements of W
⊃	X contains W	W ⊃ X
∅ or { }	Empty or null set	A = ∅
∩	Intersection or common element of two sets	A ∩ B
∪	Union of given sets, that is, all elements from the two sets	A ∪ B

W'	Complement of or not in the set	W' = { Nov, Dec, Jan } if W = {winter} then W' = {not winter}
N	Set of natural numbers	N = {1, 2, 3, ….}
Z	Set of integers	Z = { . . . . (-1), 0, (+1) . . . . }
Q	Set of rational numbers	Q = { . . . . . $(-1\frac{1}{3})$, (–0.6), $\frac{1}{2}$, 3, 14, ... }

Let us look at a few examples of what we did in Level 1 before we look into the Set Builder Notation.

Worked Example [1]

Questions

State the meaning of the following

1) $A \not\subset B$ where A= {numbers} B= {vowels}
2) $2 \notin A$ where A= {odd number}.

Solutions

1) A is not subset of B
2) 2 is not an element of set A

In order for us to understand the Set Builder Notation, we should be acquainted to the various ways in which information about sets is written down. Let us look into some few examples

Worked Example [2]

Questions

Given that $x \in N$, list the members of the sets below:

a) $\{x: x < 8\}$ b) $\{x: x \geq 4\}$ c) $\{x: x \leq 10\}$

Solutions

Remember $x \in N$ means that x is an element of natural members. Natural numbers are counting numbers from 1 to infinite (1, 2, 3…).

a. Therefore $\{x: x < 8\}$ requires you to list counting numbers less than 8. In this case x is representing counting numbers:{1, 2, 3, 4, 5, 6, 7,}

b. $\{x: x \geq 4\}$ this is a list of numbers, Thus{4, 5, 6, 7 …}. x is representing natural numbers from 4 to infinite where the three (3) dots (…) mean do not end.

$$\{x: x \leq 10\} = \{1, 2, 3, 4, 5, 6, 7, 8, 9\ 10\} \text{ Or } \{1, 2, 3, \ldots, 8, 9, 10\}$$

This is used when the elements (members) of the set are many but end. List the first three, put three dots and the last three. Another form of set builder notation is: a∈Z, meaning **a** is an element of set of integers.

TIP: Set of Z what is it? Check key words in Table 3.1.

Let us look at some more examples, and this time the set information is in a form of an equation.

Worked Example [3]

Questions

Given $a \in Z$, write down the elements of:

1) $a \in \{a: 3a + 4 \leq 10\}$

2) $a \in \{a : 16 < 3a + 1\}$

TIP: Solve the equation first and list elements.

Solutions

Z is a set of integers, integers are:

$\{\ldots -3, -2, -1, 0, 1, 2\ldots\}$

Therefore

1). $a \in \{a: 3a + 4 \leq 10\}$

$3a + 4 \leq 10$

$3a \leq 10 - 4$

$3a \leq 6$

$a \leq 2$

$a = \{2, 1, 0, -1 \ \ldots\}$

2). $a \in \{a: 16 < 3a + 1\}$

$16 < 3a + 1$

$16 - 1 < 3a$

$15 < 3a$

$5 < a$ or $a > 5$

$a = \{6, 7, 8, 9 \ \ldots\}$

Worked Example [4]

Questions

If $X \in Z$, $X = \{x: x > -3\}$ and $Y = \{x: x < 2\}$

List the elements $X \cap Y$.

TIP: You need to first identify what is in set X and set Y then choose common numbers or elements.

Solutions

$X \cap Y = \{x: x > -3\} \cap \{x: x < 2\}$

From $X \in Z$, $X = \{-2, -1, 0, 1, 2 \ldots\}$ and $Y = \{1, 0, -1, -2, \ldots\}$

Then $X \cap Y = \{-1, -2, 0, 1\}$ these are the common elements of the two sets.

Now that we have looked at some few examples, attempt the following activity.

Activity (*3.1*) Set Builder Notation

Questions

1. If $X = \{x : x \text{ is an integer}, 1 < x \leq 16\}$

   $Y = \{\text{even numbers}\}$

   $Z = \{\text{perfect square}\}$

   a) List elements of set Y.

   b) List $Y \cap Z$.

2. $\xi = \{x: 2 \leq x \leq 12, x \in N\}$,

   $A = \{x: x \text{ is a factor of } 16\}$

   $B = \{x: x \text{ is a prime number}\}$

   a) List the elements of set A.

   b) Find $n(A \cup B)$.

3. Given $\xi = \{1, 2, 3 \ldots 18, 19, 20\}$. List the elements of each set:

   a) $\{x: x \text{ is an odd number}, x \in \xi\}$.

   b) $\{x: x \text{ is a multiple of } 3, x \in \xi\}$.

   c) $\{x: x - 6 > 10, x \in \xi\}$.

   d) $\{(x; y): y > 14 + x, x \in \xi, y \in \xi\}$.

   e) $\{(y; x): 4x^2 + 2, y \in \xi, x \in \xi\}$.

   f) $\{(x; y): y \text{ is a cube of } x, 4 \leq y \leq 20, 2 \leq x \leq 8\}$.

Answers

1 (a) {2;4;6;8;10;12;14;16} (b) {4;16}

2 (a) {2;4;8} (c) 7

3 (a){1;3;5;7;9;11;13;15;17;19}

(b){3;6;9;12;15;18}

(c) {17;18;19;20}

(d) {(1;16), (1;17), (1;18), (1;19), (1;20), (2;17), (2;18), (2;19), (2;20), (3;18), (3;19), (3;20), (4;19), (4;20), (5;20)}

(e) {(1;6), (2;18)}

(f) {(2;8)}

3.3 VENN DIAGRAMS WITH 2 SUBSETS

Venn diagrams can be drawn to represent any number of sets. Let us look at examples of questions which involve drawing of Venn diagrams with 2 subsets.

Worked Example [5]

Questions

Out of 100 families 32 had radios and 51 had TV sets, if 40 had none of the two. How many had both?

Solutions

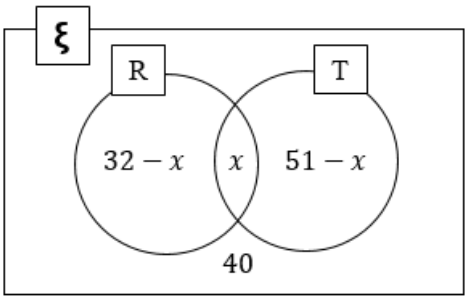
TIP: Let x represent families with both radio and TVset.

Let $\xi = \{100 \text{ families}\}$

$R = \{\text{families with radios}\}$

$T = \{\text{families with TV sets}\}$

It is required to calculate $n(R \cap T)$



Since $n(R) = 32 - x$

Also $n(T) = 51 - x, n(T \cap R) = x$ and $n(R' \cap T') = 40$

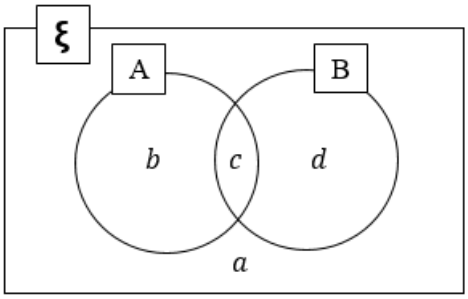
Therefore $40 + (32 - x) + x + (51 - x) = 100$

$$40 + 32 + 51 - x + x - x = 100$$
$$123 - x = 100$$
$$123 - 100 = x$$
$$23 = x$$

Therefore 23 families have both radio and TV set.

Worked Example [6]

Questions



In the above Venn diagram,

ξ = {farmers}

A = {animal rearing}

B = {crop farming}

The letters a, b, c, d represent numbers of farmers in each subsets.

Given $n(\xi) = 254, n(A) = 40$ and $n(B) = 30$

a) Express b in terms of c.

b) Express d in terms of c.

c) State the greatest possible value of c.

d) Find the smallest possible value of a.

e) Find a, b, c given that $c = 4$.

Solutions

a) b + c = 40

b = 40 – c

b) d + c = 30

d = 30 – c

c) c = 30

d) 254 – (40 + 30)

254 – 70

184 (where c = 0)

e) $(b - c) + c + (d - c) + a = 254$

i. From (a)

$b = 40 - c$

$b = 40 - 4$

$b = 36$

Therefore $b = 36$

ii. From (b)

$d = 30 - c$

$d = 30 - 4$

$d = 26$

Therefore $d = 26$

iii. $a + b + c + d = 254$

$a + 36 + 4 + 26 = 254$

$a + 66 = 254$

$a = 254 - 66$

$a = 188$

Therefore $a = 188$

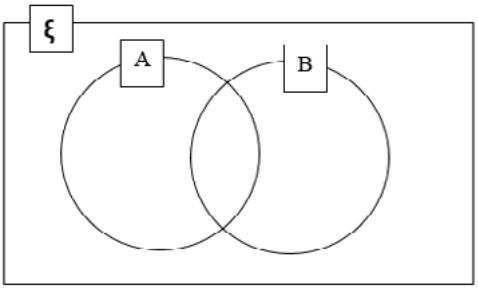
Now that we have gone through a few examples, you can now attempt the questions on the following activity.

Activity (*3.2*) Venn Diagrams

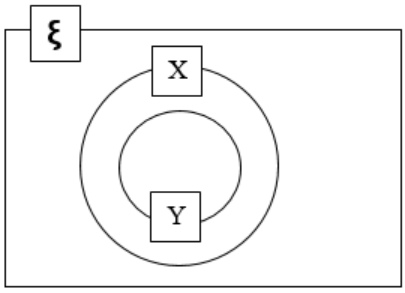
Questions

1) Shade the following Sets in the diagrams below

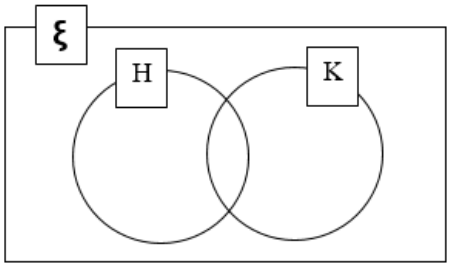
a) A ∩ B



b) X ∪ Y'



c) H ∪ K



2) List the elements of the following sets :

a) $\{x : x \geq 6, x \in N\}$

b) $\{x : x \leq 4, x \in Z\}$

c) $\{x : x < 8, x \in Z\}$

d) $\{x : x > -3, x \in Z\}$

3) Given $\xi = \{0, 1, 2, 3, \ldots . 7, 8, 9\}$, list the elements :

a. $\{x : x = 2x + 3 \leq 9, x \in \xi\}$

b. $\{(x, y) : y = x + 4, x \in \xi, y \in \xi\}$

4) $\xi = \{ x : 1 \leq x < 15, x \text{ is an integer} \}$

$A = \{ x : x \text{ is a multiple of } 3 \}$

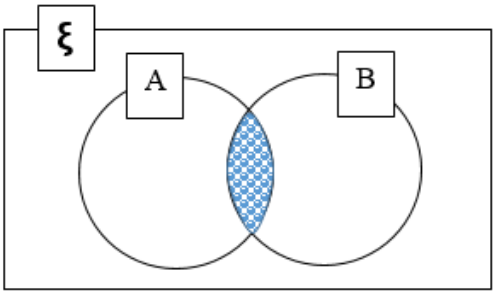
$B = \{ x : x \text{ even numbers} \}$

List members of each set:

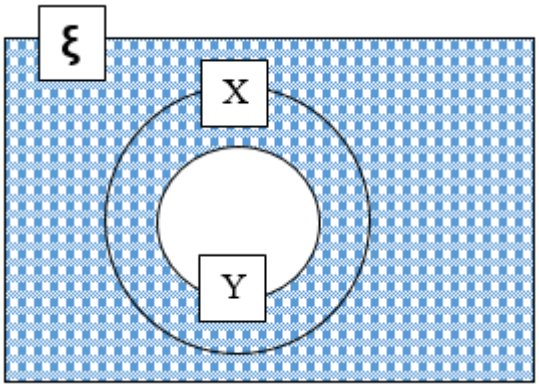
a) $A = \{ \}$ b) $B = \{ \}$ c) $A \cap B = \{ \}$

Answers

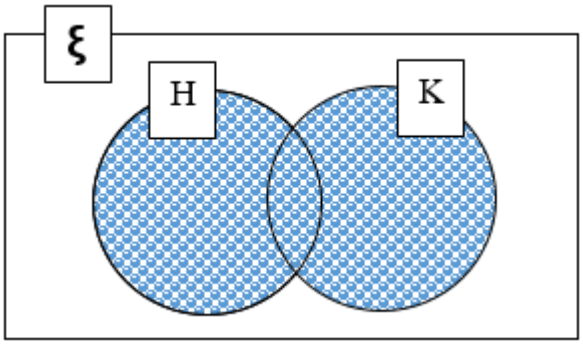
a) $A \cap B$



b) $X \cup Y'$



c) $H \cup K$



2)

a) { 6, 7, 8, ... } b) { . . . . , 2, 3, 4 }

c) { . . . ., 5, 6, 7 }

d) { -2, -1, 0 , ... }

3)

a) { 0, 1, 2, 3 }

b) { 0,4 }{ 1,5 }{ 2,6 }{ 3,7 }{ 4,8 }{ 5,9 }

4)

a) A = { 3, 6, 9, 12 }

b) B = { 2, 4, 6, 8, 10, 12, 14 }

c) A ∩ B = { 6, 12 }

3.4 VENN DIAGRAMS WITH 3 SUBSETS

It is very important to identify each set in ξ

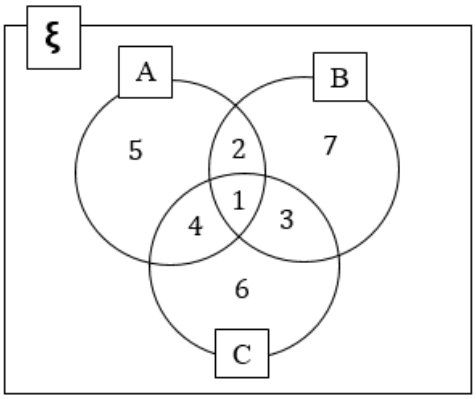


Fig 3.1

Regions:	1 = A ∩ B ∩ C	all 3 sets
	2 = A ∩ B	A & B ONLY
	3 = B ∩ C	B & C ONLY
	4 = A ∩ C	A & C ONLY
	5	A only
	6	C only
	7	B only

Once you have grasped the different regions in a set, and you are able to place information in the correct set, then it should be easy for you to solve problems involving sets.

Worked Example [7]

Question

At a school there are 10 boys. 6 study History, 5 study Bible and 7 study Maths, 3 study History and Bible, 2 study Bible and Maths and 4 study History and Maths. Each boy does at least one of the three subjects. How many boys do all three subjects? Using first letter of each subject to group the boys:

That is, $H = \{\text{boys who study History}\}$

$B = \{\text{boys who study Bible}\}$

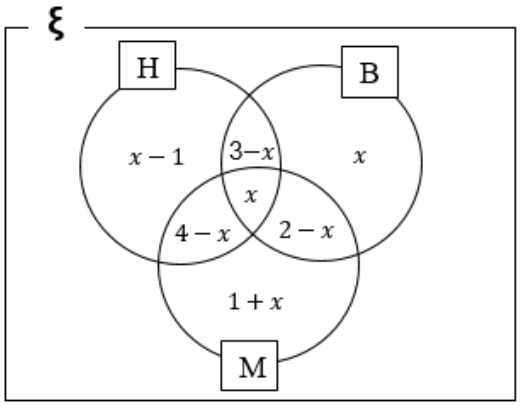
$M = \{\text{boys who study Maths}\}$

All 10 boys form your (ξ) universal set.

That is $\xi = \{\text{all ten boys}\}$

Solutions

Let x be boys how do all 3 subjects



At the centre that is where you put x meaning all the three subjects. History and Bible ($H \cap B$) is where you place $3 - x$, History and Maths ($B \cap M$) you place $2 - x$ andBible and Maths ($H \cap M$), you place $4 - x$.

Those doing History only are:

$6 - (3 - x) - x - (4 - x)$ which simplifies to:

$6 - 3 + x - x - 4 + x$

$6 - 7 + x$, this means: $x - 1$,

Also those doing Bible only are:

$5 - (3 - x) - x - (2 - x)$

$5 - 3 + x - x - 2 + x$

$5 - 5 + x$, which is x

Lastly those doing Maths only are:

$7 - (4 - x) - x - (2 - x)$

$7 - 4 + x - x - 2 + x$

$7 - 6 + x$, which is $1 + x$

TIP: Take note of boys who do one subject are worked out first. Now all sets or subsets are filled with appropriate information.

Hence:

$(x - 1) + (3 - x) + x + x + (2 - x) + (x + 1) + (4 - x) = 10$

(since $H \cup B \cup M = \xi$)

Simplifying our equation and solving for x thus:

$x - 1 + 3 - x + x + x + 2 - x + x + 1 + 4 - x = 10$

$4x - 3x - 1 + 10 = 10$

$x + 9 = 10$

Therefore $x = 1$

Meaning only one (1) boy does all three subjects.

Now that we have gone through a few examples, you should now be in a good position to attempt the following activity.

Activity (*3.3*) Venn Diagrams with 3 Subsets

Questions

1). In the diagram below, identify by shading the given set:

a) $P \cap Q \cap R$.

b) $P \cup Q \cup R$.

c) $(P \cup Q) \cap R$.

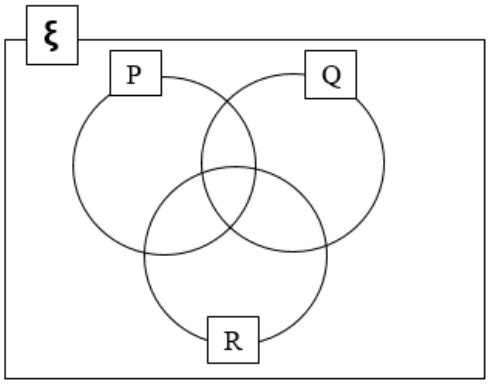
d) $(P \cap Q) \cup R$.

e) $P \cup (Q \cap R)$.

f) $Q' \cap (P \cup R)$.

g) $P' \cup (Q \cap R)$.

h) $Q' \cap (P \cap R)$



2). If in a group of learners, 24 play Handball, 28 play Baseball and 21 play Softball. 8 play Handball only, 14 play Baseball only and 7 play Handball and Baseball only and 4 play Baseball and Softball only.

How many play:

a) All three games?

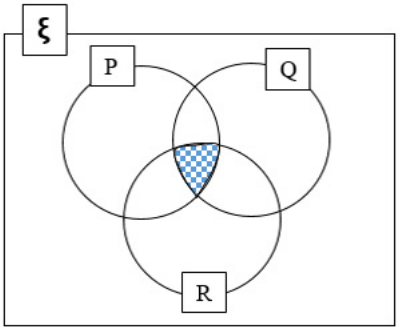
b) Handball and Softball only?

c) Softball only?

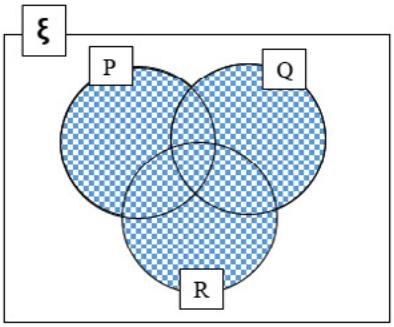
d) If 10 play none, how many learners are at the school?

Answers

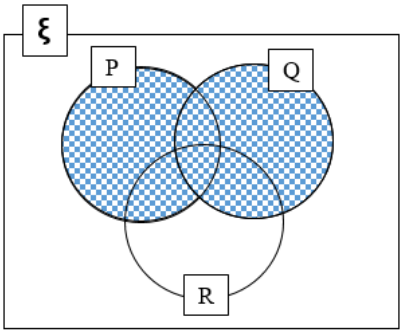
1) a) $P \cap Q \cap R$.



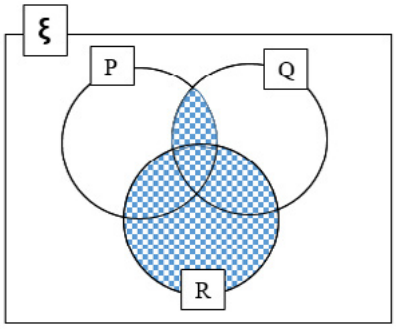
b) $P \cup Q \cup R$



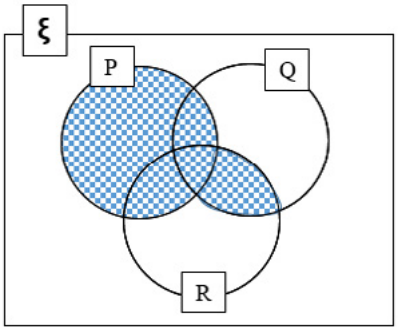
c) $(P \cup Q)$



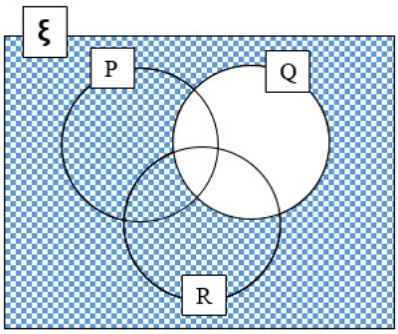
d) $(P \cap Q) \cup R$



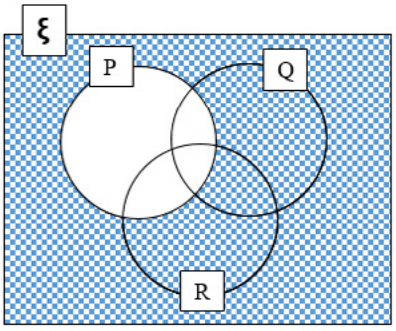
e) $P \cup (Q \cap R)$



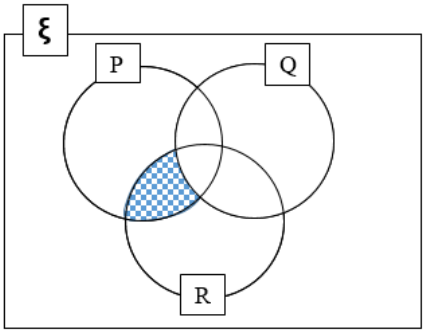
f) $Q' \cup (P \cap R)$



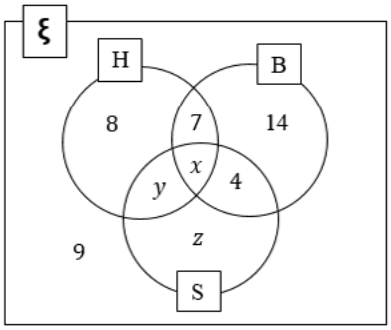
g) $P' \cup (Q \cap R)$



h) $Q' \cap (P \cap R)$



2).



In the diagram:

ξ = {all learners}

H = {Handball}

B = {Baseball}

S = {Softball}

TIP: Start by finding x since those who play baseball = 28

a. Hence: $x = 28 - (7 + 14 + 4)$

$x = 28 - 25$

$x = 3$

Therefore 3 play all the three games.

b. y represent those who play handball and softball. Since those who play handball are 24 you can find y, thus :

$y + 8 + x + 7 = 24$

$y + 8 + 3 + 7 = 24$

$y + 18 = 24$

$y = 24 - 18$

$y = 6$

Therefore 6 play handball and softball only.

c. If those who play softball are 21 then:

$x + 4 + y + z = 21$

$3 + 4 + 6 + z = 21$

$13 + z = 21$

$z = 21 - 13$

$z = 8$

Therefore those who play softball only are 8.

d. $8 + 7 + 14 + 4 + 3 + 6 + 8 + 10 = 60$

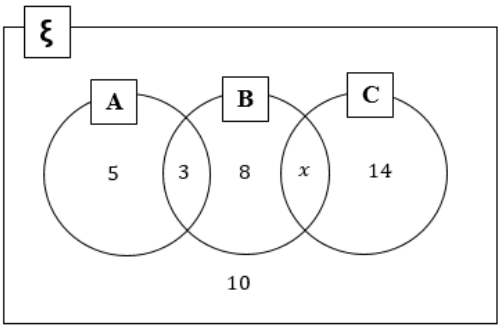
Therefore the total number of learners is 60.

Activity (*3.4*) Solving problems involving sets

Questions

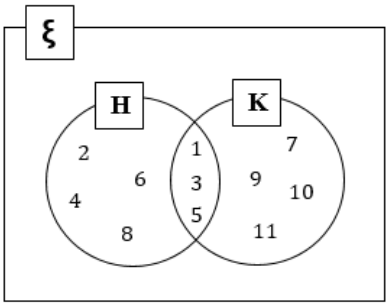
1) The diagram below shows elements in their appropriate regions. If given $n(\xi) = 42$, calculate the following:

   a) X.
   b) $n(B)$.
   c) $n(A \cup C)$.
   d) $n(A \cap C)$.
   e) $n(C' \cap B')$.
   f) $n(A' \cup B)$.



2) The Venn diagram below shows sets H and K. State:

   a) $n(H \cap K)$
   b) $n(H \cup K)$
   c) H'
   d) $(H \cup K)'$



3) In a group of drivers, 40 like fish, 31 like beef and 45 like chicken. 10 like fish only, 15 like beef only and 20 like chicken and fish. If 14 like fish and beef, and 6 like beef and chicken.

   a) Calculate how many like:
      i. Chicken only?
      ii. All three dishes?
   b) How many drivers were there?

Solutions

1. a) $10 + 5 + 3 + 8 + x + 14 = 42$

   $40 + x = 42$

   $x = 42 - 40$

   $x = 2$

   b) $n(B) = 3 + 8 + 2$

   $= 13$

   c) $n(A \cup C) = (5 + 3) + (2 + 14)$

   $= 8 + 16$

   $= 24$

   d) $n(A \cap C) = 0$[there are no common elements for set A and set C i.e they are disjoint sets].

   e) $n(C' \cap B') = 5 + 10$

   $= 15$

   f) $n(A' \cup B) = 8 + 10 + 14 + 2$

   $= 34$

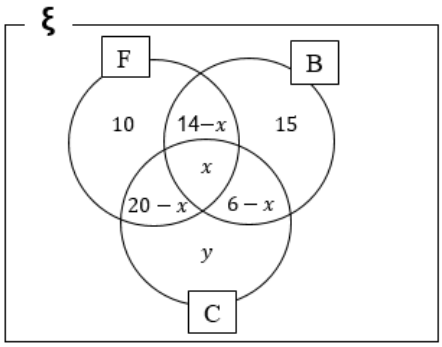
2. a) $n(H \cap K) = 3$

   b) $n(H \cup K) = 11$

   c) $H' = \{7, 9, 10, 11\}$

   d) $(H \cup K)' = \{\ \}$[there are no elements outside the two sets This is a null set]

3.



In the diagram let:

F = {drivers who like fish}

B = {drivers who like beef}

C = {drivers who like chicken}

TIP: note the word **only** this will help you to identify sets as shown in the diagram above, and let those who like chicken only be y. those who like fish and beef may like chicken too, this is why 14 – x and those, who like beef and chicken might like fish.

a. You should start by calculating x taking those who like beef or fish:

$$\text{Beef} \Rightarrow (14 - x) + x + (6 - x) + 15 = 31$$
$$14 - x + x + 6 - x + 15 = 31$$
$$15 + 14 + 6 - x = 31$$
$$35 - x = 31$$
$$x = 35 - 31$$
$$x = 4$$

$$\text{or Fish} \Rightarrow 10 + (14 - x) + x + (20 - x) = 40$$
$$10 + 14 - x + x + 20 - x = 40$$
$$44 - x = 40$$
$$x = 44 - 40$$
$$x = 4$$

i) Therefore those who like all three dishes are 4.

ii) Those who like chicken only thus:

$$(20 - x) + x + (6 - x) + y = 45$$
$$20 - x + x + 6 - x + y = 45$$
$$20 - 4 + 4 + 6 - 4 + y = 45$$

$20 + 6 - 4 + y = 45$

$22 + y = 45$

$y = 45 - 22$

$y = 23$

Therefore 23 drivers like chicken only.

b. $10 + 10 + 4 + 16 + 2 + 15 + 23 = 80$

Therefore the total numbers of drivers is 80.

3.5 Summary

In this unit you have learnt about Sets, Set Builder Notations, Venn diagrams with three subsets and the application of Venn diagrams in solving real-life problems. Knowledge gained in this unit on Sets is very useful in solving some of our day to day challenges.

3.6 Further Reading

1. Macrae, M, Madungwe, L and Mutangadura, A (2017) New General Mathematics Book 3, Pearson Capetown.
2. Macral, M, Madungwe, L and Mutangadura, A (2017) New General Mathematics Book 4, Pearson Capetown.
3. Meyers, C; Graham, B; Dawe, L (2004) Mathscape Working Mathematically, 9th Ed, MacMillan
4. Pimentel R and Wall T (2011) International Mathematics, Hodder UK
5. Rayner D (2005) Extended Mathematics. Oxford New York

3.7 Assessment Test

1) Given $\xi = \{x: 0 \leq x < 100, x \text{ is an integer}\}$

$M = \{x: x \text{ is a multiple of } 4\}$

$P = \{x: x \text{ is a perfect square}\}$

a) List the elements of M. [2]

b) List the elements of P. [2]

c) Find $n(M \cap P)$. [1]

d) List the elements of $(M \cup P)$. [1]

e) Find $n(M \cup P)$. [1]

2) If given $\xi = \{x: x < 15, x \in Z\}$ (consider positive integers only)

A = {even number}

B = {odd number}

C = {3, 6, 9, 12}

Find:

a) n (B ∩ C) [2]

b) A' [2]

c) (A ∩ C)' [2]

d) (B ∪ C) [2]

3) If $\xi = \{1, 2, 3, \ldots 18, 19, 20\}$, list the members of the following sets:

a) $\{x: x + 2 > 16\}$ [2]

b) $\{(x, y): y = 15 + x, y \in \xi, x \in \xi\}$ [3]

c) $\{(x, y): y = 3x^2 - 5, x \in \xi, y \in \xi\}$ [3]

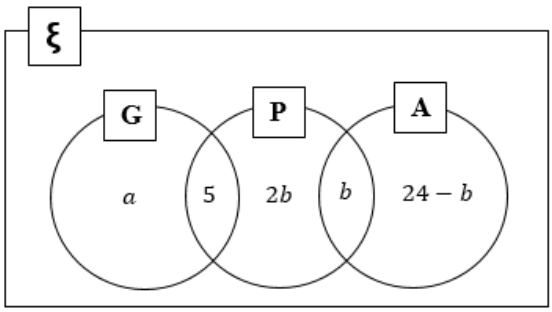
4) 54 girls chose a drink each with at least one of the following flavours:

Guava(G).

Peach (P).

Apple (A).

The Venn diagram below represents the number of girls in the subsets.



a) Write down the total number of girls who chose apple.

b) Express **a** in terms of **b** in its simplest form.

c) Girls who chose apple only were 3 more than those whose who chose peach only.

i. Write an equation in b.

ii. Solve your equation.

iii. Calculate the number of girls who chose peach only.

Answers

1). a) M = {4, 8, 12, 16, 20, 24, 28, 32, 36, 40, 44, 48, 52, 56, 60, 64, 68, 72, 76, 80, 84, 88, 92, 96}

b) P = {,1 4, 9, 16, 25, 36, 49, 64, 81}

c) 4

d) {4, 8, 9, 12, 16, 20, 24, 25, 28, 32, 36, 40, 42, 44, 48, 49, 52, 56, 60, 64, 68, 72, 76, 80, 81, 84, 84, 88, 92, 96}

e) 31

2). a) 2 b) [1, 3, 5, 7, 9, 11, 13] c) {1, 2, 3, 4, 5, 7, 8, 9, 10, 11, 13, 14}

d) [1, 3, 5, 6, 7, 9, 11, 12, 13]

3). a) {15, 16, 17, 18, 19, 20} b) {(1, 16), (2, 17), (3, 18), (4, 19), (5, 20)}

c) (2, 7)

4). a) 24 b) a = 25 – b c(i) 2b + 3 = 24 - b (ii) b = 7 (iii) 14

UNIT 4 - ALGEBRA 2

CONTENTS

4.1 INTRODUCTION

This unit is a sequel to the topic on Algebra which you did in Unit 2.It seeks to equip you with the skills required in dealing with manipulation of algebraic equations. Algebraic manipulation, that is, simultaneous equations and quadratic equations. You will realise that these skills are essential in solving numerous real-life problems.

OBJECTIVES

After going through this unit you should be able to:

- solve different types of equations
- change the subject of a given formula
- substitute in a given formula
- solve both simultaneous and quadratic equations using different methods.
- use the quadratic equation formula to solve quadratic equations.

KEY TERMS:

Equation: - is an algebraic statement that contains an equal sign.

Solve: - find the value of the unknown

Formula: - a statement, especially an equation used to express a mathematical rule or principle.

Linear equation: – it is a mathematical statement with an equal sign in which the highest power of the unknown is 1. For example 2x +4 = 6 where x is the unknown and its power is 1.

Subject of the formula: – it is a variable which is expressed in terms of other variables involved in the formula.

Unknown: – a letter in an equation, representing a number.

Variables: – letters that are used to represent numbers.

TIME: You should not spend more than 10 hours on this unit.

STUDY SKILLS

You should have completed the unit on Algebra 1 for you to undertake this unit with ease. The key to understanding algebra is practice. You are advised to do as many algebraic questions as possible for you to fully grasp all the concepts involved.

4.2 LINEAR EQUATIONS

It is a mathematical statement with an equal sign in which the highest power of the unknown is 1. For example $2x + 4 = 6$ where x is the unknown and its power is 1. There are basically two main classifications of linear equations. Some equations are regarded as simple linear equations while others are regarded as linear equations with fractions.

4.2.1 Simple Linear Equations

Linear equations are solved by collecting like terms together on the same side of an equal sign.

Worked Example [1]

Question

If there are three people, Musa, Kuda and John. Let us say Musa is x years old, Kuda is 5 years older than Musa, John is 3 times older than Kuda and the sum of their ages is 100.

Find the ages of Musa, Kuda and John.

Solution

To find their ages you have to formulate an equation as shown below:

Let Musa's age be x, Kuda's age= $\mathrm{x}+5$ and John's age will be $=3(\mathrm{x}+5)$

Sum of their ages is

$$\mathrm{x}+\mathrm{x}+5+(3\mathrm{x}+15)$$
$$2\mathrm{x}+5+3\mathrm{x}+15$$
$$5\mathrm{x}+20$$

If the sum of ages is 100, then $5\mathrm{x}+20=100$.

$5\mathrm{x}+20=100$

$5\mathrm{x}+20-20=100-20$

$5\mathrm{x}=80$

$\frac{5\mathrm{x}}{5}=\frac{80}{5}$

Therefore $\mathrm{x}=16$

Musa's age= x

$=16$

Kuda's age $=\mathrm{x}+5$

$=16+5$

$=21$

John's age= $3(x + 5)$

$= 3(16 + 5)$

$= 3(21)$

$= 63$

Now, let us look at some more examples of linear equations.

Worked Example [2]

Questions

Solve the following linear equations

1) $5 - 2x = 7$
2) $3x - 5 = 7 - 4x$
3) $2m + 3 = 18 - 3m$
4) $4(3x - 2) - 3(2x - 1) = 7$
5) $6 - 4(2x - 3) = 2$

Solutions

1) $5 - 2x = 7$

$-2x = 7 - 5$ [Collecting like terms]

$-2x = 2$

$\frac{-2x}{-2} = \frac{2}{-2}$ [Dividing both sides by -2]

$x = -1$

2) $3x - 5 = 7 - 4x$

$3x + 4x = 7 + 5$ [Collecting like terms]

$7x = 12$

$\frac{7x}{7} = \frac{12}{7}$ [Dividing both sides by 7]

$x = 1\frac{5}{7}$

3) $2m + 3 = 18 - 3m$

$2m + 3m = 18 - 3$ [Collecting like terms]

$5m = 15$

$\frac{5m}{5} = \frac{15}{5}$ [Dividing both sides by 5]

$m = 3$

4) $4(3x - 2) - 3(2x - 1) = 7$

$12x - 8 - 6x + 3 = 7$

$6x - 5 = 7$

$6x = 7 + 5$

$6x = 12$

$\frac{6x}{6} = \frac{12}{6}$

$x = 2$

5) $6 - 4(2x - 3) = 2$

$6 - 8x + 12 = 2$

$-8x + 18 = 2$

$-8x = 2 - 18$

$-8x = -16$

$\frac{-8x}{-8} = \frac{-16}{-8}$

$x = 2$

TIP: You can check the solution of your equation by substituting the value of the unknown in the equation. Check weather $x = 2$ in question 4.

4) $4(3x - 2) - 3(2x - 1) = 7$

$4(3 \times 2) - 2) - 3(2 \times 2) - 1) = 7$

$4(6 - 2) - 3(4 - 1) = 7$

$4(4) - 3(3) = 7$

$16 - 9 = 7$

$7 = 7$

L.H.S = R.H.S, therefore the value of x is correct. Now that you have gone through the examples on how to solve linear equations, attempt the following activity.

Activity (*4.1*) Addition and subtraction

Questions

Solve the following equations

1) $3x + 8 = 12 - 2x$
2) $2m + 12 - 5m = 7m - 8$
3) $3\frac{1}{2}m = 49$
4) $12 - (3b + 5) = 16$
5) $3(3x - 2) - 4(5x - 1) = 0$

Answers

1) $x = \frac{4}{5}$
2) $m = 2$
3) $m = 14$
4) $b = -3$
5) $x = \frac{2}{-11}$

4.2.2 Equations with fractions

When solving equations with fractions you first need to clear the fractions by multiplying throughout by a common denominator (L.C.M) as shown in the following examples.

Worked Example [3]

Questions

Solve the following equations

1) $\frac{2x+1}{3} + \frac{x-4}{5} = 2$
2) $\frac{4x-1}{4} - 3\frac{1}{2} = \frac{2x+3}{6}$

Solutions

1) $\frac{2x+1}{3} + \frac{x-4}{5} = 2$

2) $\frac{2x+1}{3} + \frac{x-4}{5} = \frac{2}{1}$

Find the LCM of 3, 5, 1 which is 15 and multiply each term by 15

$$\frac{15(2x+1)}{3}+\frac{15(x-4)}{5}=\frac{15(2)}{1}$$

Simplify

$$5(2x+1)+3(x-4)=30$$

Remove brackets

$$10x+5+3x-12=30$$

Collect like terms

$$10x+3x=30+12-5$$
$$13x-7=30$$
$$13x=30+7$$
$$13x=37$$
$$\frac{13x}{13}=\frac{37}{13}$$
$$x=\frac{37}{13}$$
$$x=2\frac{11}{13}$$

3) $\frac{4x-1}{4}-3\frac{1}{2}=\frac{2x+3}{6}$

Change $3\frac{1}{2}$ to an improper fraction.

$$\frac{4x-1}{4}-\frac{7}{2}=\frac{2x+3}{6}$$

Find the LCM of 2, 4, 6 which is 12 and multiply each term by 12

$$\frac{12(4x-1)}{4}-\frac{12(7)}{2}=\frac{12(2x+3)}{6}$$

Simplify

$$3(4x-1)-6(7)=2(2x+3)$$

Remove brackets

$12x - 3 - 42 = 4x + 6$

Collect like terms

$12x - 4x = 6 + 3 + 42$

$8x = 51$

$\frac{8x}{8} = \frac{51}{8}$

$x = \frac{51}{8}$

$x = 6\frac{3}{8}$

Now that you have gone through the examples, you can now attempt the following activity.

Activity (*4.1*) Equations with Fractions

Questions

1. Solve the following equations

a) $\frac{9x}{10} + \frac{2}{5} = \frac{2x}{5} + \frac{3}{10}$

b) $\frac{3x+4}{4} + \frac{x}{3} = \frac{2x-1}{4}$

c) $\frac{4}{5}(2x - 1) = \frac{2}{3}(2x+5) + 1\frac{1}{3}$

d) $\frac{x}{3} + 1\frac{1}{4}x + \frac{5}{12} = \frac{2x+1}{4}$

e) $\frac{4a-1}{3} - \frac{3a-1}{2} = \frac{a}{4}$

2. Formulate and solve equations on the following word problems

a) Two farmers sold the same number of bags of maize. If farmer A harvested 120 bags and farmer B harvested 80 bags of maize. If the number of bags left by farmer A are twice the number of bags left by farmer B. Calculate the number of bags each sold.

b) The sum of 8 and $\frac{3}{5}$of xproduces a result which is thrice x. Find the value of x.

c) Peter and John shared \$240. Peter got \$30 more than John. Calculate John's share.

Answers

1) (a) $\frac{-1}{5}$ (b) $-2\frac{1}{7}$ (c) $20\frac{1}{2}$ (d) $\frac{-2}{13}$ (e) $\frac{2}{5}$

2) (a) 40 (b) $3\frac{1}{3}$ (c) \$105

4.3 SUBSTITUTION OF VALUES

Substitution refers to putting values where the letters are. For example the formula for area of a trapezium = $\frac{1}{2}$(a+b)h. It can be written as $A = \frac{1}{2}(a+b)\,h$ where A (area) and a and b are parallel sides, then h the height of the trapezium. To find A the values of a, b and h should be given so that we substitute them in the formula. As shown in the examples below.

Worked Example [4]

Questions

Given that $A = \frac{1}{2}(a+b)\,h$, find A when $a = 5$cm, $b = 11$cm and $h=6$cm.

Solution

$A = \frac{1}{2}(5+11)6$ (after substituting)

$A = \frac{1}{2} \times 16 \times 6$

$A = 48\text{cm}^2$

Given a formula with unknown quantities, the other given quantities can be used to find the unknown quantity.

Worked Example [5]

Questions

If $A = \frac{1}{2}bh$, find A when b = 8cm and h=10cm

Solutions

$A = \frac{1}{2} \times 8 \times 10$ substitute b with 8 and h. with 10

$= 40cm^2$

There are instances where you may be given $A = 30cm^2$ and h = 10cm, and then tasked to find b. consider the following example.

Worked Example [6]

Question

If $A = \frac{1}{2}(a+b)\ h$.Find b when $A = 30cm^2$ and h = 10cm

Solution

$30 = \frac{1}{2} \times b \times 10$ (substituting in the formula)

$30 = 5b$

$\frac{30}{5} = \frac{5b}{5}$

b = 6cm

Worked Example [7]

Questions

Given that $I = \frac{P \times R \times T}{100}$

a) Find I when P = \$560, R = 15% and T = 5 years

b) Find R when I=\$450, P=\$750 and T=1.5 years

Solutions

a) $I = \frac{P \times R \times T}{100}$[substitute P, R and T]

$I = \frac{560 \times 15 \times 5}{100}$

I=$420

b) $I = \frac{P \times R \times T}{100}$[substitute I, P and T]

$450 = \frac{750 \times R \times 1.5}{100}$

$450 = 750 \times R \times 1.5$

$45000 = 1125R$

$\frac{45000}{1125} = \frac{1125R}{1125}$

R=40

Activity (*4.3*) Substitution

Questions

1) Given that $A = \frac{1}{2}(a+b)\ h$
   a) Find A, given that a=7, b=5 and h=8
   b) Find h, given that A= 40, a= 8 and b=12
2) Given that $C=2\pi r$
   a) Find C, given that $\pi = \frac{22}{7}$ and r =14
   b) Find r, given that $\pi = \frac{22}{7}$ and C = 121
3) When calculating the wages of his workers an employer uses the formula $W = 250 + \frac{2T}{5}$ where W is the wage and T is overtime in hours.
   a) Calculate the wage(W) if the employee had overtime(T) of 20hrs.
   b) Calculate overtime if the employee got $360 as wages.

Answers

1. (a) 48	(b) 42.
2. (a) 88	(b) 19.25
3.(a) 258	(b) 275

4.4 CHANGING THE SUBJECT OF THE FORMULA

Subject of the formula refers to a variable which is expressed in terms of other variables involved in the formula. For example, in the formula $y = ax + b$, 'y' is the subject, in order to make either, a, or x or b the subject, the formula has to be re-arranged, as shown in the following examples.

Worked Example [8]

Questions

Make x the subject in the following equations

1) $y = ax + b$
2) $y = x + a$
3) $y = x - a$
4) $y = ax$
5) $y = ax + bx$

Solutions

1) $y = ax + b$

[b is adding ax so when moved to left, it changes the sign]

$y - b = ax$

[since a is multiplying x, to remove a we divide both sides by a to make x the subject]

$$\frac{y - b}{a} = x$$

$$x = \frac{y - b}{a}$$

2) $y = x + a$

$y - a = x$

$x = y - a$

3) $y = x - a$

$y + a = x$

4) $y = ax$

$\frac{y}{a} = x$

$x = \frac{y}{a}$

5) $y = ax + bx$

$y = x(a + b)$

4.4.1 Formula with brackets

When you want to change the subject of the formula where there are brackets, first remove the brackets. As shown below.

Worked Example [9]

Questions

Make x the subject of the following

$y = a(2 - x)$

Solution

$y = a(2 - x)$

$y = 2a - ax$

$y + ax = 2a$

$ax = 2a - y$

$x = \frac{2a - y}{a}$

4.4.2 Formula with squares and square roots

When you want to change the subject of the formula where there is a square root, first remove the square root by squaring both sides.

When you want to change the subject of the formula where a quantity is squared, first remove the square by finding the square root on both sides. Consider the following examples.

Worked Example [10]

Questions

Make x the subject in the following

1) $y = x^2$
2) $y = \sqrt{x}$
3) $y = a\sqrt{x}$
4) $y = \sqrt{ax}$
5) $y = a\sqrt{(x + b)}$

Solutions

1.) $y = x^2$

[to remove the square we find the square root on both sides]

$\sqrt{y} = \sqrt{x^2}$

$\sqrt{y} = x$

Therefore $x = \sqrt{y}$

2.) $y = \sqrt{x}$

$y^2 = (\sqrt{x})^2$

[square both sides]

$y^2 = x$

Therefore $x = y^2$

1) $y = a\sqrt{x}$

$\frac{y}{a} = \frac{a\sqrt{x}}{a}$

[divide both sides by a]

$\frac{y}{a} = \sqrt{x}$

$(\frac{y}{a})^2 = (\sqrt{x})^2$

[square both sides to remove the square]

$\frac{y^2}{a^2} = x$

Therefore $x = \frac{y^2}{a^2}$

2) $y = \sqrt{ax}$

$y^2 = (\sqrt{ax})^2$

square both sides

$y^2 = ax$

$\frac{y^2}{a} = \frac{ax}{a}$

$\frac{y^2}{a} = x$

Therefore $x = \frac{y^2}{a}$

3) $y = a\sqrt{(x + b)}$

$\frac{y}{a} = \frac{a\sqrt{(x+b)}}{a}$

$\frac{y}{a} = \sqrt{(x + b)}$

$\frac{y^2}{a^2} = x + b$

$\frac{y^2}{a^2} - b = x$

Therefore $x = \frac{y^2}{a^2} - b$

Activity (4.4) Change of subject of formula

Questions

Make the letter in brackets the subject of the formula in each question

1) $y = mx - b$ (x)

2) $y = mx + bx$ (x)

3) $ax + y = bx - 5$ (x)

4) $y = a(a + bx)$ (x)

5) $v = \frac{\pi r^2 h}{3}$ (r)

6) $v^2 = u^2 - 2as$ (a)

7) $v = ut + \frac{1}{2}at^2$ (a)

8) $v = u + at$ (t)

9) $m = \frac{b^2+ax}{x+a}$ (x)

10) $a = 2\pi\sqrt{\frac{h}{w}}$ (w)

11) $m = \frac{b-n}{x}$ (n)

12) $I = \frac{PRT}{100}$ (t)

13) $\frac{1}{a} = \frac{1}{b} + \frac{1}{c}$ (b)

14) $M = \sqrt{(\frac{abx}{3+y})}$ (b)

15) $m = \frac{3p+5}{x}$ (p)

Answers

1) $x = \frac{y+b}{m}$

2) $x = \frac{y}{m+b}$

3) $x = \frac{y+5}{b-a}$

4) $x = \frac{y-a^2}{ab}$

5) $r = \sqrt{\frac{3v}{\pi h}}$

6) $a = \frac{u^2-v^2}{2s}$

7) $a = \frac{2(v-ut)}{t^2}$

8) $t = \frac{v-u}{a}$

9) $\frac{b^2-ma}{m+a}$

10) $w = \frac{4\pi^2 h}{a^2}$

11) $n = b - mx$

12) $T = \frac{100I}{PR}$

13) $b = \frac{ac}{c-a}$

14) $b = \frac{m^2(3+y)}{ax}$

15) $\frac{mx-5}{3}$

Hint: The expression 'in terms of' can be used in order for one to change the subject of the formula.

Worked Example [11]

Questions

1) Given that $y = ax - b$ write x in terms of y, a and b.
2) Given that $a = b\sqrt{(c+1)}$, express c in terms a and b.

Solution

1) Given that $y = ax - b$ write x in terms of y, a and b.

   $y = ax - b$

   [write x in terms of y, a and b means make x the subject of the formula.]

   $\frac{y+b}{a} = \frac{ax}{a}$

   $x = \frac{y+b}{a}$

2) Given that $a = b\sqrt{(c+1)}$, express c in terms a and b.

   $a = b\sqrt{(c+1)}$

   $\frac{a}{b} = b\sqrt{(\frac{c+1}{b})}$ divide both sides by b

   $\frac{a}{b} = \sqrt{(c+1)}$

   $\frac{a^2}{b^2} = c+1$

   $\frac{a^2}{b^2} - 1 = c$

   Therefore $c = \frac{a^2}{b^2} - 1$

4.5 SIMULTANEOUS EQUATIONS

Simultaneous equations are equations that have two unknowns for example,

$2x + 3y = 6$

$x - y = -2$

The unknowns in the above simultaneous equations are x and y.

Some of the methods used to solve simultaneous equations are elimination and substitution.

4.5.1 Solving simultaneous equations by elimination

This method involves eliminating one of the unknown by either subtracting or adding the two equations. Let us consider the following examples.

Worked Example [12]

Questions

Solve the following simultaneous equation by finding the values of x and y which satisfy both equations

1) $x + y = 6$
   $x - y = 2$

2) $2x + 3y = 6$
   $x - y = -2$

Solutions

1) $x + y = 6 \quad (1)$
   $x - y = 2 (2)$

   To eliminate the variable y, we add equations (1) + (2)

   $2x = 8$

   $x = 4$

   To find the value of y, we substitute x = 4 into either equation (1) or (2)

Substituting x = 4 into equation (1):

$4 + y = 6$

$y = 6 - 4$

$y = 2$

To check that the solution is correct, the values of x and y are substituted into either equation (1) or (2). If the solution is correct, then the left hand side of the chosen equation will be equal to the right hand side.

Using equation (2)

$x - y = 2$

$4 - 2 = 2$

$2 = 2$

Therefore x = 4, y = 2

2) $2x + 3y = 6$ (1)

$x - y = -2$ (2)

To eliminate the variable x, we multiply equation (2) by 2

$2x + 3y = 6$

$2x - 2y = -4$

Then subtract equation (1) – (2)

$5y = 10$

$y = 2$

To find the value of x, we substitute y = 2 into either equation (1) or (2)

Substituting y = 2 into equation (1):

$x - y = -2$

$x - 2 = -2$

$x = -2 + 2$

$x = 0$

To check that the solution is correct using equation (2)

$x - y = -2$

$0 - 2 = -2$

$-2 = -2$

Therefore $x = 0$, $y = 2$

4.5.2 Solving Simultaneous Equations by Substitution

This method involves rearranging the equation to make the unknown the subject.

Worked Example [13]

Questions

Solve the following simultaneous equation by finding the values of x and y which satisfy both equations

$2x + y = 5$ (1)

$3x - 2y = 4$ (2)

Solution

$2x + y = 5$ (1)

$3x - 2y = 4$ (2)

Equation (1) can be rearranged to give:

$y = 5 - 2x$

hence this can be substituted into equation (2)

$3x - 2(5 - 2x) = 4$

$3x - 10 + 4x = 4$

$3x + 4x = 4 + 10$

$7x = 14$

$x = 2$

To find the value of y, x=2 is substituted into either equation (1) or (2):

$2x + y = 5$

$2\ (2) + y = 5$

$4 + y = 5$

$y = 5 - 4$

$y = 1$

To check that the solution is correct using equation (1)

$2x + y = 5$

$2\ (\ 2) + 1 = 5$

$4 + 1 = 5$

$5 = 5$

Therefore $x = 2$, $y = 1$

Activity (*4.5*) Simultaneous Equations

Questions

Solve the following simultaneous equations using a suitable method.

1) $4a + b = 14$
   $a + 5b = 13$

2) $x + 2y = 8$
   $2x + 3y = 14$

3) $3c - d = 5$
   $2c + 5d = 9$

4) $3r + 2w = 2$
   $r + 5w = -8$

Answers

1) $a = 3$
   $b = 2$

2) $x = 4$
   $y = 2$

3) $c = 2$
   $d = 1$

4) $r = 2$
   $w = -2$

4.6 QUADRATIC EQUATIONS

A quadratic equation is an equation whereby the highest power of the unknown is 2, that is, the highest power of the variable x is x^2. The following are types of quadratic equations.

$x^2 - 3x - 10 = 0$. $2y^2 - 13y + 15 = 0$. $m^2 - 8 = 0$. $3r^2+4r+1=0$.

4.6.1 Solving quadratic equations by factorisation

Consider the following examples on how quadratic equations can be solved using the factorisation method.

Worked Example [14]

Questions

Solve the following equations by factorising

1) $x^2 - 5x = 0$
2) $x^2 + 10x + 25 = 0$
3) $x^2 + 2x - 24 = 0$
4) $2x^2 + 11x + 12 = 0$
5) $3x^2 + 7x - 6 = 0$

Solution

1) $x^2 - 5x = 0$

First you factorise

$x(x - 5) = 0$

so either $x = 0$ or $x - 5 = 0$

$x = 0$ or $x = 5$

2) $x^2 + 10x + 25 = 0$

We need to find two values which multiply to give $+25x^2$ and which add to give $10x$.

The only two values which satisfy both conditions are $+5x$ and $+5x$.

$x^2 + 5x + 5x + 25 = 0$

$x(x + 5) + 5(x + 5) = 0$

Therefore $(x + 5)(x + 5) = 0$

So $x + 5 = 0$

$x = -5$ twice

3) $x^2 + 2x - 24 = 0$

We need to find two values which multiply to give $-24x^2$ and which add to give $2x$.

The only two values which satisfy both conditions are $+6x$ and $-4x$.

$x^2 - 4x + 6x - 24 = 0$

$x(x - 4) + 6(x - 4) = 0$

Therefore $(x + 6)(x - 4) = 0$

So either $x + 6 = 0$ or $x - 4 = 0$

$x = -6$ or $x = 4$

4) $2x^2 + 11x + 12 = 0$

We need to find two values which multiply to give $24x^2$ ($2x \times 12x$) and which add to give $11x$.

The only two values which satisfy both conditions are $+8x$ and $+3x$.

$2x^2 + 8x + 3x + 12 = 0$

$2x(x + 4) + 3(x + 4) = 0$

Therefore $(x + 4)(2x + 3) = 0$

So either $x + 4 = 0$ or $2x + 3 = 0$

$x = -4$ or $x = \frac{-3}{2}$

5) $3x^2 + 7x - 6 = 0$

We need to find two values which multiply to give $-18x^2$ (3×-6) and which add to give $7x$.

The only two values which satisfy both conditions are $+9x$ and $-2x$.

$3x^2 + 9x - 2x - 6 = 0$

$3x(x + 3) - 2(x + 3) = 0$

Therefore $(x + 3)(3x - 2) = 0$

So either $x + 3 = 0$ or $3x - 2 = 0$

$x = -3$ or $x = \frac{2}{3}$

Now that you have gone through the examples, you can now attempt the following activity.

Activity (*4.6*) Quadratic Equations

Questions

Solve the following quadratic equations

1) $x^2 + 7x + 10 = 0$
2) $a^2 - 8a + 12 = 0$
3) $2x^2 - 3x - 2 = 0$
4) $3b^2 + 10b - 8 = 0$
5) $x^2 - 36 = 0$
6) $5m^2 - 2m = 0$
7) $x^2 + 6x + 9 = 0$
8) $2x^2 + 3x + 1 = 0$
9) $3x^2 - 27 = 0$
10) $m^2 = 8m$

Answers

1) –2 or –5
2) 2 or 6
3) 2 or $\frac{-1}{2}$
4) –4 or $\frac{2}{3}$
5) –6 or 6
6) 0 or $\frac{2}{5}$
7) –3 twice
8) –1 or $-\frac{1}{2}$
9) –3 or 3
10) 0 or 8

4.6.2 Solving Quadratic Equations using the Perfect Square Case.

Remember, if you are solving equations such as $x^2 = 9$, you make 'x' the subject of the formula.

$x^2 = 9$

$\sqrt{x^2} = \sqrt{9}$

$x = \sqrt{9}$

$x = \pm 3$

$x = 3$ or -3

Another type of equation $(x+3)^2 = 49$ can be solved using the same method.

$(x+3)^2 = 49.$

$\sqrt{(x+3)^2} = \pm\sqrt{49}$

$x = -3 \pm 7$

so either $x = -3+7$ or $x = -3-7$

therefore $x=4$ or $x = -10$

Perfect squares

What is a perfect square? This is a number which can be written perfectly to the power of 2. For example $4 = 2^2$ or $9 = 3^2$. This is also applicable in quadratic expressions which are perfect squares for example $x^2 + 6x + 9 = (x + 3)^2$.

Numbers such as 5 and 7 cannot be written perfectly to the power of 2. To write such numbers as 5 and 7 perfectly to the power 2, we express the number partly as a perfect square that is

$5 = 4 + 1$

$= 2^2 + 1$

Where the quadratic is not a perfect square we make an effort to express it partly as a perfect square for example

$x^2 + 4x + 7 = x^2 + 4x + 2^2 - 2^2 + 7 = (x^2 + 4x + 2^2) - 2^2 + 7$

Now this means $x^2 + 4x + 7 = (x + 2)^2 + 3$

The following examples will explain how to put quadratic expression as a perfect square

Worked Example [15]

Questions

Make the following expression perfect squares

1) $a^2 + 8a$ 2) $x^2 - 10x$

Solutions

1) $a^2 + 8a$

Add a constant obtained by dividing the coefficient of 'a' by 2 and then square the result

$= a^2 + 8a + (\frac{8}{2})^2$

$= a^2 + 8a + (4)^2$

$= a^2 + 8a + 16$

Now we have to factorise by looking for two values which multiply to give 16 and add to give 8a. The numbers are +4 and +4.

$= a^2 + 4a + 4a + 16$

$= a(a + 4)\ 4(a + 4)$

$= (a + 4)\ (a + 4)$

$= (a + 4)^2$

2) $x^2 - 10x$

Add a constant obtained by dividing the coefficient of 'x' by 2 and then square the result

$= x^2 - 10x + (\frac{-10}{2})^2$

$= x^2 - 10x + (-5)^2$

$= x^2 - 10x + 25$

Now we have to factorise by looking for two values which multiply to give 25 and add to give –10x. The numbers are –5 and –5.

$= x^2 - 5x - 5x + 25$

$= x(x-5) - 5(x-5)$

$= (x - 5)\ (x - 5)$

$= (x - 5)^2$

Worked Example [16]

Questions

Using the method of completing the square solve

1) $a^2 + 6a + 7 = 0$
2) $x^2 + 3x - 2 = 0$

Solutions

1) $a^2+6a+7=0$

Take 7 to the right side of the equation

$a^2 + 6a = -7$

Add the square of half the coefficient of 'a' on both sides

$a^2 + 6a + (\frac{6}{2})^2 = -7 + (\frac{6}{2})^2$

$a^2 + 6a + (3)^2 = -7 + (3)^2$

$a^2 + 6a + 9 = -7 + 9$

$a^2 + 6a + 9 = 2$

Factorise the expression $a^2 + 6a + 9$ by looking for two values which multiply to give 9 and add to give +6a. The numbers are +3 and +3.

$a^2 + 3a + 3a + 9$

$a(a + 3)\ 3(a + 3)$

The result produced is a perfect square $(a + 3)^2$. Right now you have to use the perfect square to complete the equation,

$a^2 + 6a + 9 = 2$

$(a + 3)^2 = 2$

$\sqrt{(a + 3)^2} = \pm\sqrt{2}$

$(a + 3) = \pm\sqrt{2}$

$a = -3 \pm\sqrt{2}$

So either $a = -3 + \sqrt{2}$ or $a = -3 - \sqrt{2}$

2) $x^2+3x -2=0$

$x^2+ 3x = 2$

$x^2+ 3x + (\frac{3}{2})^2 = 2 + (\frac{3}{2})^2$

$x^2+ 3x + \frac{9}{4} = 2 + \frac{9}{4}$

$(x + \frac{3}{2})^2 = \frac{17}{4}$

$\sqrt{(x + \frac{3}{2})^2} = \pm\frac{\sqrt{17}}{4}$

$x+ \frac{3}{2} = \pm\frac{\sqrt{17}}{2}$

$x = -\frac{3}{2} \pm \frac{\sqrt{17}}{2}$

so either $x = \frac{-3+\sqrt{17}}{2}$ or $x = \frac{-3-\sqrt{17}}{2}$

TIP: Usually the method of completing the square is done in equations that you cannot factorise.

Activity (4.7) Completing the square

Questions

Use the method of completing the square to solve the following equations

1) $x^2 + 2x - 2=0$
2) $x^2 - 8x + 1 = 0$
3) $a^2 + 5a + 2 = 0$
4) $m^2 - 5m + 2 = 0$
5) $p^2 + 8p + 15 = 0$

Answers

1) $-1\pm\sqrt{3}$
2) $4\pm\sqrt{15}$
3) $\frac{-5\pm\sqrt{17}}{2}$
4) $\frac{5\pm\sqrt{17}}{2}$5)
5) –3 or –5

4.6.3 Solving Quadratic Equations using the Quadratic Formula

A quadratic equation takes the form $ax^2+bx+c=0$, where a, b and c are whole numbers. Quadratic equations can also be solved by the use of the quadratic formula which states that:

$x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$. the following examples shows how to use the quadratic formula

Worked Example [17]

Questions

Solve the following equations using the quadratic formula. Give your answer to correct two decimal places.

1) $3x^2 - 13x - 10 = 0$
2) $x^2 + 7x + 3 = 0$

Solutions

1) $3x^2 - 13x - 10 = 0$,

a = 3, b = –13 , c = –10

substituting these values in the quadratic formula gives:

$$x = \frac{-(-13) \pm \sqrt{-13^2 - 4 \times 3 \times -10}}{2 \times 3}$$

[Hint; the division should cover everything on the numerator]

$$x = \frac{13 \pm \sqrt{169 - (-120)}}{6}$$

$$x = \frac{13 \pm \sqrt{169 + 120)}}{6}$$

$$x = \frac{13 \pm \sqrt{289}}{6}$$

$$x = \frac{13 \pm 17}{6}$$

So either $x = \frac{13+17}{6}$ or $x = \frac{13-17}{6}$

$x = \frac{30}{6}$ or $x = \frac{-4}{6}$

$x = 5$ or $x = -0.67$

2) $x^2 + 7x + 3 = 0$

a = 1, b = 7, c = 3

Substituting these values in the quadratic formula gives:

$$x = \frac{-7 \pm \sqrt{7^2 - 4 \times 1 \times 3}}{2 \times 1}$$

$$x = \frac{-7 \pm \sqrt{49 - 12}}{2}$$

$$x = \frac{-7 \pm \sqrt{37}}{2}$$

$$x = \frac{-7 \pm 6.08}{2}$$

So either $x = \frac{-7+6.08}{2}$ or $x = \frac{-7-6.08}{2}$

$x = \frac{-0.92}{2}$ or $x = \frac{-13.08}{2}$

$x = -0.46$ or $x = -6.54$

Now that you have gone through the examples of solving a quadratic equation using the quadratic formula, attempt the following activity.

Activity (*4.8*) Quadratic formula

Questions

Solve the following equations using the quadratic formula. Give your answers correct to 2 decimal places where necessary.

1) $2x^2+5x+3 =0$
2) $5x^2- 3x - 2=0$
3) $3x^2- 8x + 2 = 0$
4) $4x^2+7x - 2=0$
5) $5x^2+8x - 2=0$

Answers

1) -1 or $\frac{-3}{2}$ 2) 1 or $\frac{-2}{5}$ 3) 2,3or0,28 4) -2 or $\frac{1}{4}$ 5) 0,22or –1,82

4.7 Summary

In this unit, you have acquired the necessary skills that can help you in changing the subject of given formula, substitute in a given formula, solve both simultaneous and quadratic equations using different methods. Use these skills acquired to solve problems that you meet in other units. After going through the assessment test below, move on to the next unit.

4.8 Further Reading

1. Macrae, M, Madungwe, L and Mutangadura, A (2017) New General Mathematics Book 3, Pearson Capetown.

2. Macral, M, Madungwe, L and Mutangadura, A (2017) New General Mathematics Book 4, Pearson Capetown.
3. Meyers, C; Graham, B; Dawe, L (2004) Mathscape Working Mathematically, 9th Ed, MacMillan
4. Pimentel R and Wall T (2011) International Mathematics, Hodder UK
5. Rayner D (2005) Extended Mathematics. Oxford New York

4.9 Assessment Test

1) (a) Solve $4(2x-3)=4$ (2)

   (b) Given that $a(bx-c)=y$,make x the subject of the formula. (3)

2) (a) Solve the equation $\frac{3x-8}{5}=2$ (2)

(d) If $\frac{ax+b}{c}=y$ write x in terms of a, b, c and y (3)

3) A bank uses the formula $S = a(1+\frac{b}{100})$ to calculate salaries of its workers.

   a) Calculate S when a = 900 ,b = 300 (2)

   b) Make b the subject of the formula. (3)

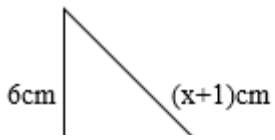
4) Solve the equations

   a) $3x^2+8x+4=0$ (2)

   b) $\frac{a-1}{3}+\frac{2a-3}{4}=\frac{5}{6}$ (3)

5) Solve the quadratic equation $x^2+6x-2=0$ leaving answer to 2 decimal places (5)

6) (a) By connecting the 3 sides of the right-angled triangle below form an equation in terms of x in its simplest form. (2)



   (b) Hence find the value of x. (3)

Answers

1) (a) 2

(b) $\frac{y-ac}{ab}$

2) (a) 6

(b) $x = \frac{cy-b}{a}$

3) (a) 3600

(b) $b = \frac{100s-100a}{a}$

4) (a) -2 or $\frac{-2}{3}$

(b) 2.3

5) 0.32 or 6.32

6) (a) $4x = 36$

(b) 9

UNIT 5 - FUNCTIONAL GRAPHS 1

CONTENTS

5.1 INTRODUCTION

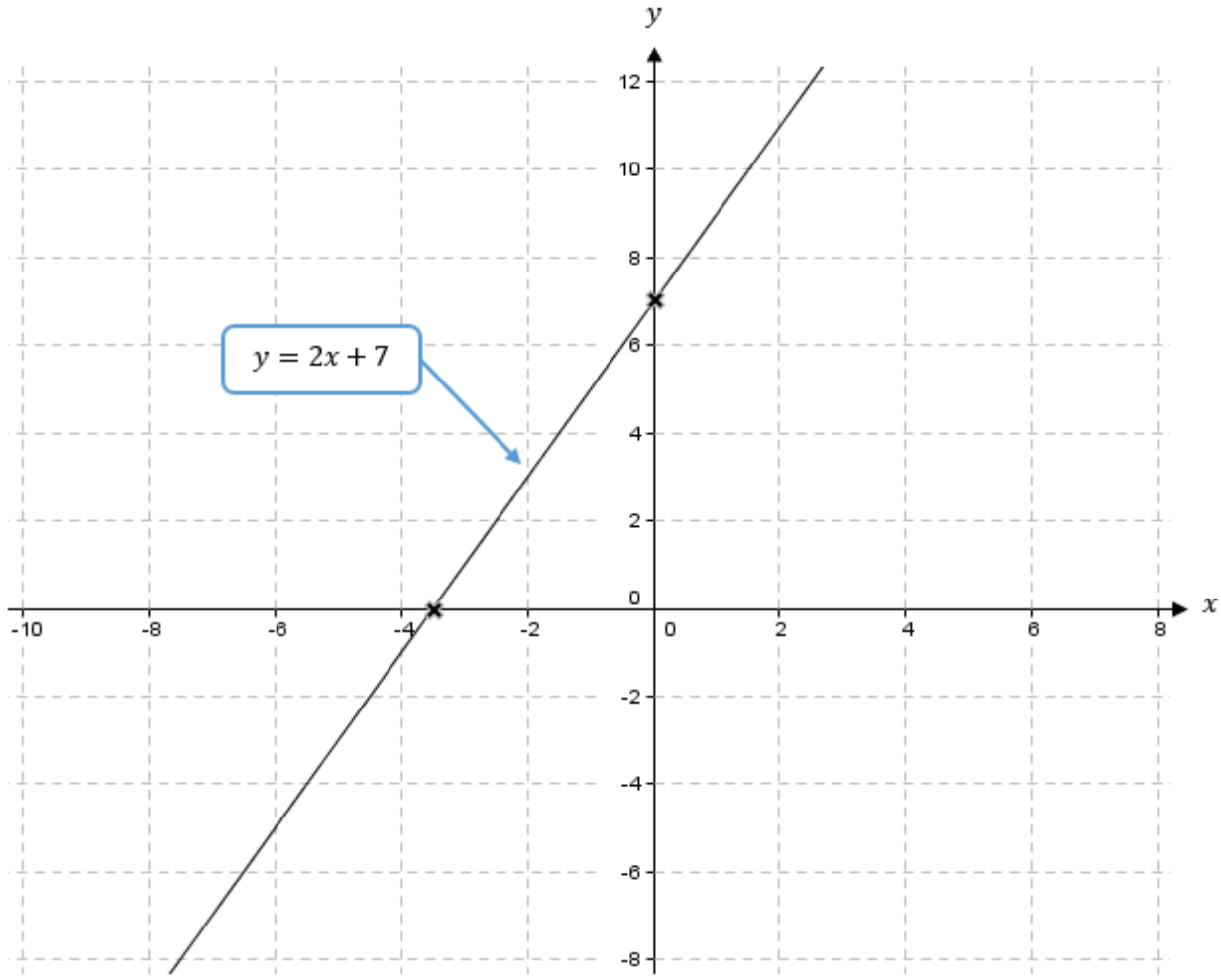


Fig 5.1

Have you ever asked yourself why do we use graphs? Well you are not alone. The answer to the questions is that graphs give a pictorial view of information which may not be easily understood on its own. The pictorial presentation may even speak volumes of information which other methods of data presentation could not

tell. In this unit you will learn how to interpret the functional notation, plot functional graphs and calculate other aspects related to the functional graphs such as area under the graph, gradients and finding roots of intersecting lines.

OBJECTIVES

After going through the unit, you should be able to:

- ✓ use and interpret the functional notation
- ✓ construct a table of values from a given function
- ✓ use given scale correctly to construct the functions
- ✓ draw graphs (functions) of the form
  - $y = mx + c$ (linear function)
  - $y = x^3$ (cubic function)
- ✓ estimate the area under the curve
- ✓ draw tangents at given points
- ✓ estimate the gradient of lines and curves at given points
- ✓ solve equations graphically
- ✓ use graphs to determine the maximum and minimum turning points

KEY WORDS

The following are key words and their meaning in mathematics

Tangent – this is a straight line which passes through a point on a curve, it just touches a curve at a point

Gradient - this is the measure of the steepness of a line.

Line of symmetry – the line which divide a shape (a curve in our case) into two equal parts

Intersection- is the point where two lines or curves meet or join.

Maximum- this is the highest of greatest point reached

Minimum - this is the least value of the smallest value reached

TIME

You are advised not to spend more than 10 hours in this unit.

STUDY SKILLS

You should have a graph book or graph papers.

Do not proceed to the next topic before you grasp the prior concept.

5.2 FUNCTIONAL NOTATION

5.2.1 Formation of the functional notation

Suppose you are given the expression $x + 3$ and then you are told that the value of x is 2. If you replace x with 2, what will be the value of the expression? The value will be 5. $x + 3$ is a rule which connects a number to another number. $x + 3$ is the rule connecting 2 to 5. The rule is called a function. Therefore $x + 3$ is a function. The letter 'f' denotes the function. The result of applying a function to a number is denoted by $f(x)$. The above example can be expressed as $f(x) = x + 3$. The equation $y = 6x + 5$ can be written in functional notation as $f(x) = 6x + 5$. The inclusion of $f(x)$ is what we call the functional notation.

Let us look at the following examples on how to convert to functional notation

Worked Example [1]

Questions

Convert the following to functional notation

a) $y = 2x + 4$

b) $y = 4x^2 - 5x$

c) $y = \frac{3x+2}{2x+1}$

Solutions

a) In the equation $y = 2x + 4$ you simply replace 'y' with $f(x)$ it becomes $f(x) = 2x + 4$

b) In the equation $y = 4x^2 - 5x$ you simply replace again 'y' with $f(x)$ it becomes $f(x) = 4x^2 - 5x$

c) In the equation $y = \frac{3x+2}{2x+1}$ again , you replace 'y' with $f(x)$ it becomes $f(x) = \frac{3x+2}{2x+1}$

Do this yourself

Convert the following equations to function notation

a) $y = 7x - 4$

b) $5x^2 + 4x = y$

c) $y = \frac{x^2+4x}{3}$

If all this is to be done correctly, you just have to replace 'y' by $f(x)$ to convert to functional notation in all the equations.

5.2.2 Calculations in functional notation

Let us use our functions/rule to find out how we can connect numbers or to find out other numbers. Consider the following examples.

Worked Example [2]

Questions

Given that $f(x) = 2x + 9$ find

a) $f(0)$

b) $f(3)$

c) $f(-1)$

d) $f(\frac{1}{2})$

Solutions

TIP - you simply replace the value of x with the given value

a) You have to replace the x value with the '0' in the function $f(x) = 2x + 9$

$f(0) = 2(0) + 9$

$f(0) = 0 + 9$

$f(0) = 9$

b) Replace the x value with the '3' in the function $f(x) = 2x + 9$

$f(3) = 2(3) + 9$

$f(3) = 6 + 9$

$f(3) = 15$

c) Replace the x value with '-1'in the function $f(x) = 2x + 9$

$f(-1) = 2(-1) + 9$

$f(-1) = -2 + 9$

$f(-1) = 7$

d) Replace the x value with '$\frac{1}{2}$' in the function $f(x) = 2x + 9$. Work out your answer in the space provided below

Do this yourself;

Given that $f(x) = \frac{4x}{x+2}$, find

a) $f(2)$

b) $f(-1)$

TIP; you just replace the value of x with the given value

Now let us consider an example which requires us to find the value of x given the outcome of using the function

Worked Example [3]

Question

Given that $f(x) = 2x + 4$, find x if $f(x) = 10$

Solution

The question requires you to find the value of x which will result in the rule obtaining a 10. The value of x which will result in the rule obtaining a 10 is 3. Now here is how we obtain the 3

$f(x) = 2x + 4$ and $f(x) = 10$

since $f(x) = f(x)$ it also implies that $2x + 4 = 10$

now let us solve for x on the equation $2x + 4 = 10$

$2x = 10 - 4$

$2x = 6$ (divide both sides by 2)

$\frac{2x}{2} = \frac{6}{2}$

$x = 3$

Do this yourself;

Given that $f(x) = \frac{3x}{x-1}$, find x if $f(x) = 4$

You have to equate the two values of $f(x)$ and solve the equation. Use the space provided below

If this is done correctly, the value of x is 4

Now let us consider another example.

Worked Example [4]

Question

Given that $f(x) = kx + 8$, find k if $f(2) = 12$

Solution

In this question you are told that if you substitute x with 2 in the given function the result should be 12. Therefore you just have to substitute x with 2 and equate to 12

It has been given that $f(x) = kx + 8$ and that $f(2) = 12$

Now substitute x with 2 and equate to 12

$f(2) = k(2) + 8 = 12$

Now observe that $k(2)$ will have to be written as $2k$

$2k + 8 = 12$

Now solve for k

$2k + 8 = 12$

$2k = 12 - 8$

$2k = 4$ (dividing both sides by 2)

$\frac{2k}{2} = \frac{4}{2}$

$k = 2$

Do this yourself;

Given that $f(x) = mx - 15$ find k if $f(4) = 1$ find the value of m in the space below

Here you just have to substitute the value of x with 4 then you equate the whole thing to 1

If all this is done correctly, m should be 4

Activity (*5.1*) Functional Notation

Questions

1) Given that $f(x) = x + 3$ find
   a) $f(0)$
   b) $f(-2)$
   c) $f(\frac{1}{2})$

2) Given that $f(x) = x^2 - 3x - 10$, find
   a) $f(0)$
   b) $f(-1)$
   c) the value of x if $f(x) = 0$

3) If $f(x) = \frac{3x+2}{x-1}$ find
   a) $f(2)$
   b) x if $f(x) = 4$

4) Given that $f(x) = kx + 12$, find the value of k if $f(2) = 24$

5) If $f(x) = \frac{r+3x}{rx+1}$, find r if $f(-2) = 1$

6) Given that $f(x) = k^2x^2 - 3kx - 11$ find the two values of k if $f(1) = -1$

Answers

1. (a) 3
   (b) 1
   (c) 3.5

2. (a) –10
   (b) –6
   (c) x = 5 or –2

3. x = 8

4. x = 6

5. r = 2.5

6. k = 5 or –2

5.3 LINEAR FUNCTIONS/GRAPHS

What are linear functions? Linear functions are straight line graphs. The highest power of the variable is 1. Here are a few examples of linear functions

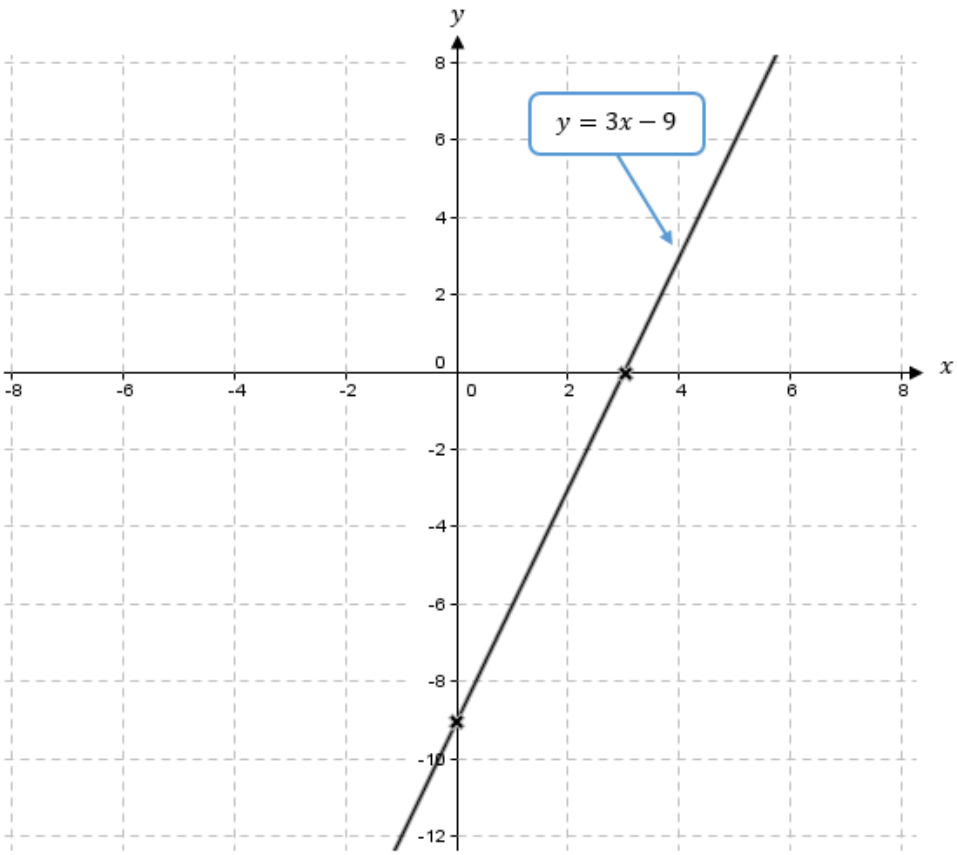


Fig 5.2

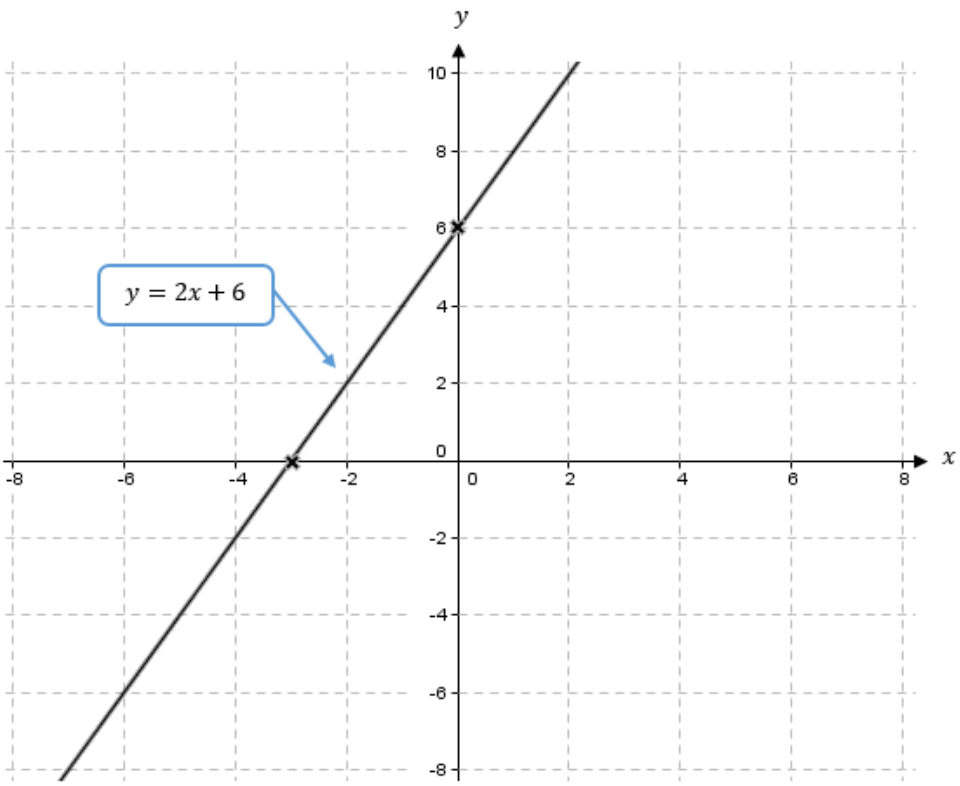


Fig 5.3

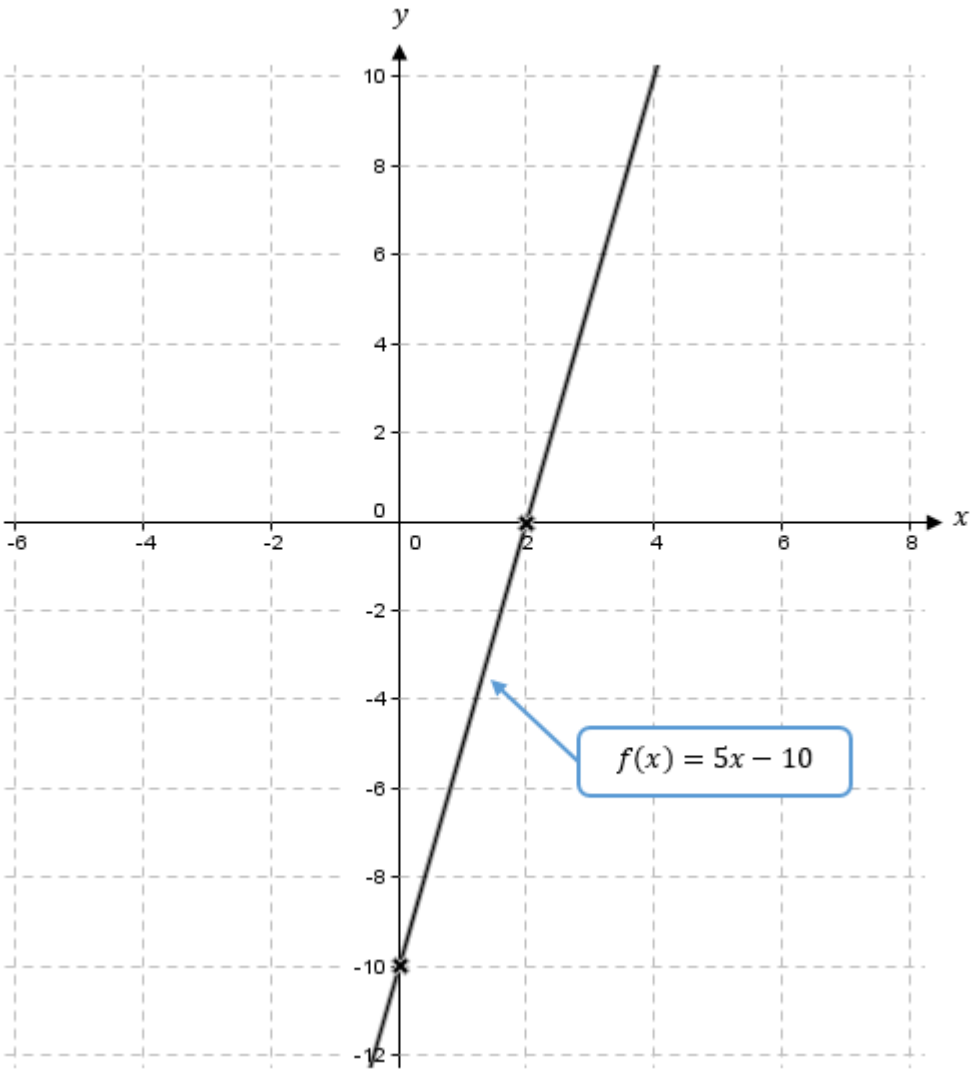


Fig 5.4

All these are example of linear functions and the highest power of the variable (x) is one. In the space below write any three examples of linear functions

5.3.1 Plotting a linear graph/function

Do you know how to plot a linear graph? If you don't know, don't trouble yourself. After going through this example below you will be well acquainted on how to plot or draw a straight line.

TIP – In order to plot a graph of this nature, you should first draw up a table of values.

Worked Example [5]

Question

Plot the graph of $y = 3x + 6$. Using a scale of 2cm to represent 1 unit on the x –axis and 2 cm to represent 2 units on the y –axis

Solution

Step 1– Draw up a table of values

x			
y			

Step 2 – Choose the x values you would want to use

x	–1	0	1
y			

Step 3 – Calculate the values of y

When $x = -1$: we find the value of y by substituting the value of x on the given linear equation. We use the x values from the table of values.

$$y = 3x + 6$$
$$y = 3(-1) + 6$$
$$y = -3 + 6$$
$$y = 3$$

x	–1	0	1
y	3		

When $x = 0$:

$y = 3x + 6$

$y = 3(0) + 6$

$y = 0 + 6$

$y = 6$

x	–1	0	1
y	3	6	

If you keep on substituting the values of x in the equation you are going to find all the values of y. Finish off the working in the space below to find the values of y when the value of $x = 1$. Check your answers with the completed table of values below.

The completed table of values is as follows:

x	–1	0	1
y	3	6	9

Step 4 – Using the given scale, plot the coordinates from the table of values as follows

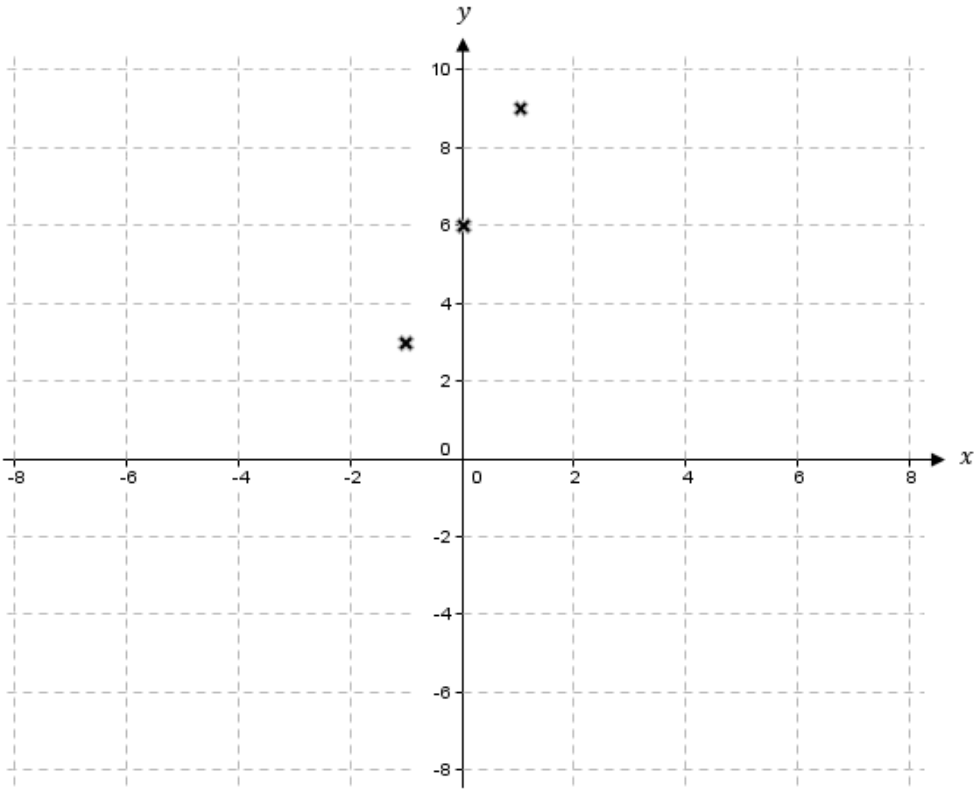


Fig 5.5

Step 5 – Join the coordinates using a ruler to produce a straight line and label the line

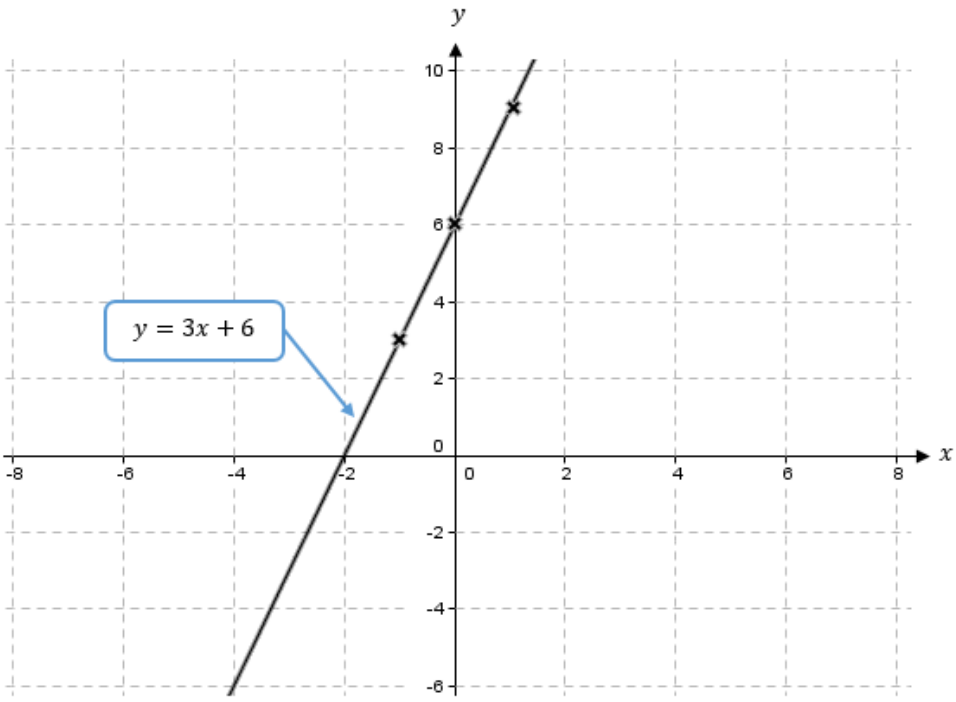


Fig 5.6

Let us try another example to plot a linear equation of the form ax + by = c

Worked Example [6]

Question

Draw the graph of the line $5x + 2y = 10$, using a scale of 2cm to represent 1 unit on both axis

Solution

Step 1 – Draw up a table of values with the x values of your choice (sometimes the x values will be provided for you)

x	–1	0	1	2
y				

Step 2 – Calculate the y values using the given equation which is $5x + 2y = 10$. You substitute the x values in the equation

When $x = -1$; the value of y will be calculated as follows

$5(-1) + 2y = 10$

$-5 + 2y = 10$

$2y = 10 + 5$

$2y = 15$ (divide both sides by 2)

$\frac{2y}{2} = \frac{15}{2}$

$y = 7\frac{1}{2}$ $(7,5)$

x	–1	0	1	2
y	7,5			

When $x = 0$; the value of y is calculated as follows (substituting the x value in the equation)

$5x + 2y = 10$

$5(0) + 2y = 10$

$0 + 2y = 10$

$2y = 10$ (divide both sides by 2)

$\frac{2y}{2} = \frac{10}{2}$

$y = 5$

x	–1	0	1	2
y	7,5	5		

Do the calculations for the values of y when $x = 1$ and when $x = 2$ in the space provided below

See if the answers you have calculated are the same as the answer provided in the table of values below

x	–1	0	1	2
y	7,5	5	2,5	0

Step 4 – Using the coordinates from the table of values, mark the coordinates on the Cartesian plane with an x as shown below

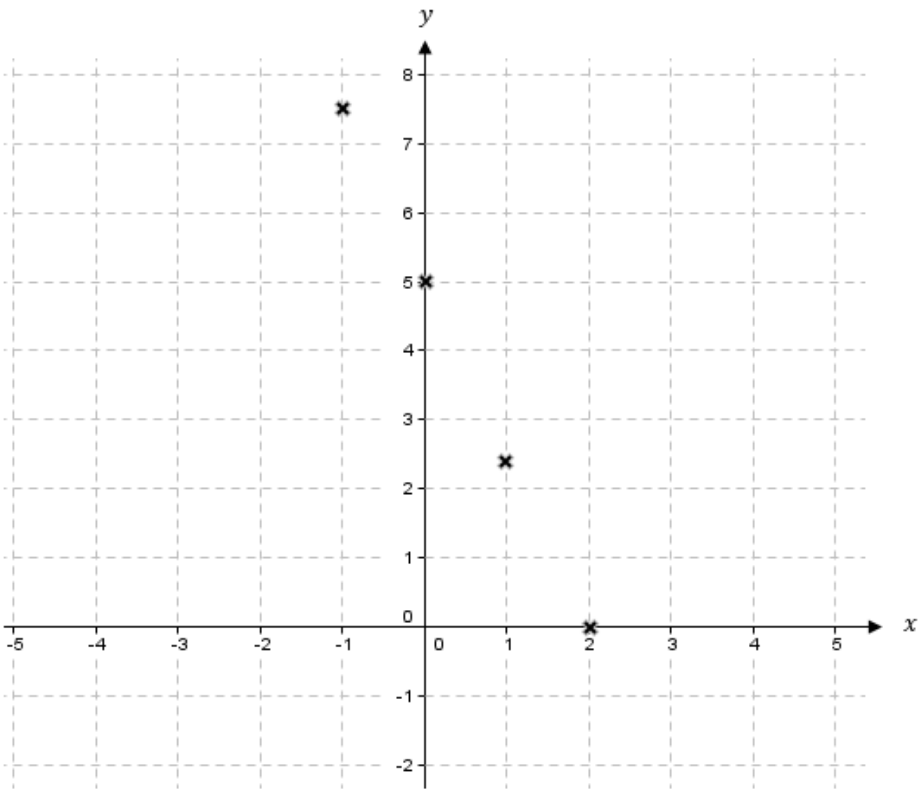


Fig 5.7

Step 5 – Join the marked coordinates with a ruler to make up a straight line and label the line.

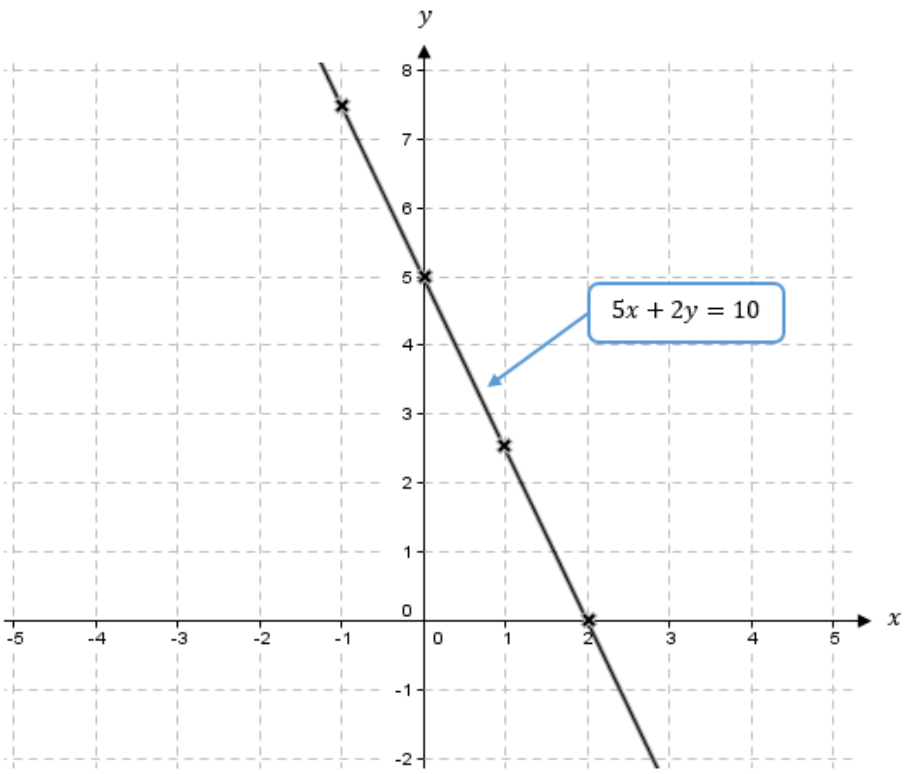


Fig 5.8

Activity (*5.2*) Line Graphs

Questions

Complete the table of values and draw the graphs for the following equations using an appropriate scale

1) $y = 2x + 1$

x			2	3
y	-1	1		

2) $y + 2x = 2$

x	–2	0		
y			0	–3

3) $5x - 2y = 5$

x	–1	0	1	2
y				

4) $y = 6x - 12$

x	–1		3	
y		–6		0

Answers

1) $y = 2x + 1$

x	–1	0	2	3
y	–1	1	5	7

2) $y + 2x = 2$

x	–2	0	1	–0,5
y	6	2	0	3

3) $5x - 2y = 5$

x	–1	0	1	2
y	–5	–2,5	0	2,5

4) $y = 6x - 12$

x	–1	1	2	3
y	–18	–6	0	6

5.3.2 Gradient of linear functions

What do you understand about the concept of gradient? Gradient is defined as the rate of change in y compared to the change in x. Gradient can either be positive or negative. The following graphs shows lines which have positive and those which have negative gradient.

Positive Gradient

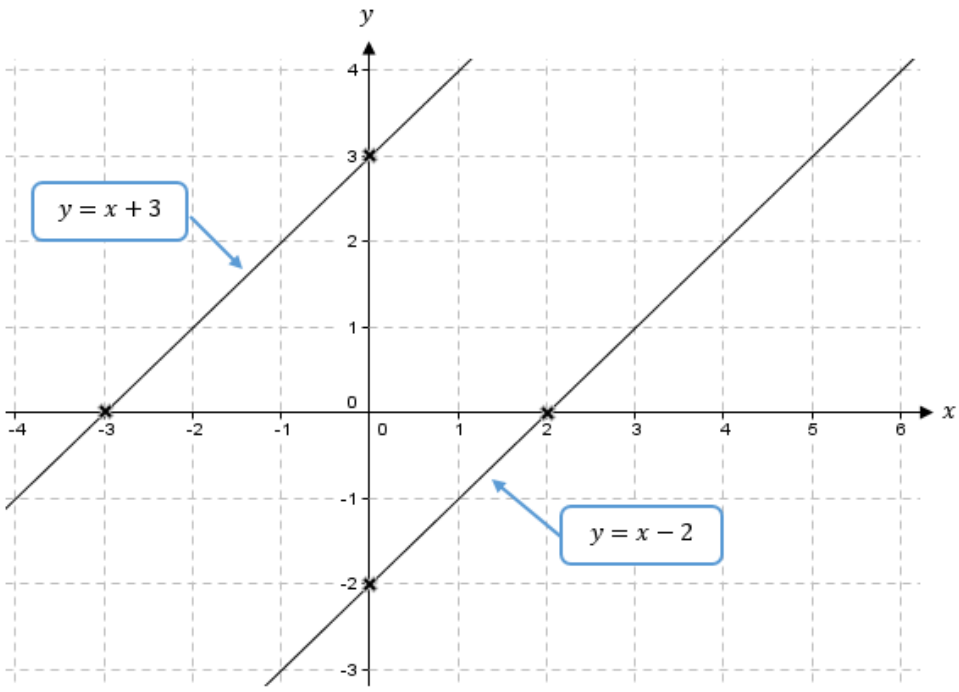


Fig 5.9

Negative Gradient

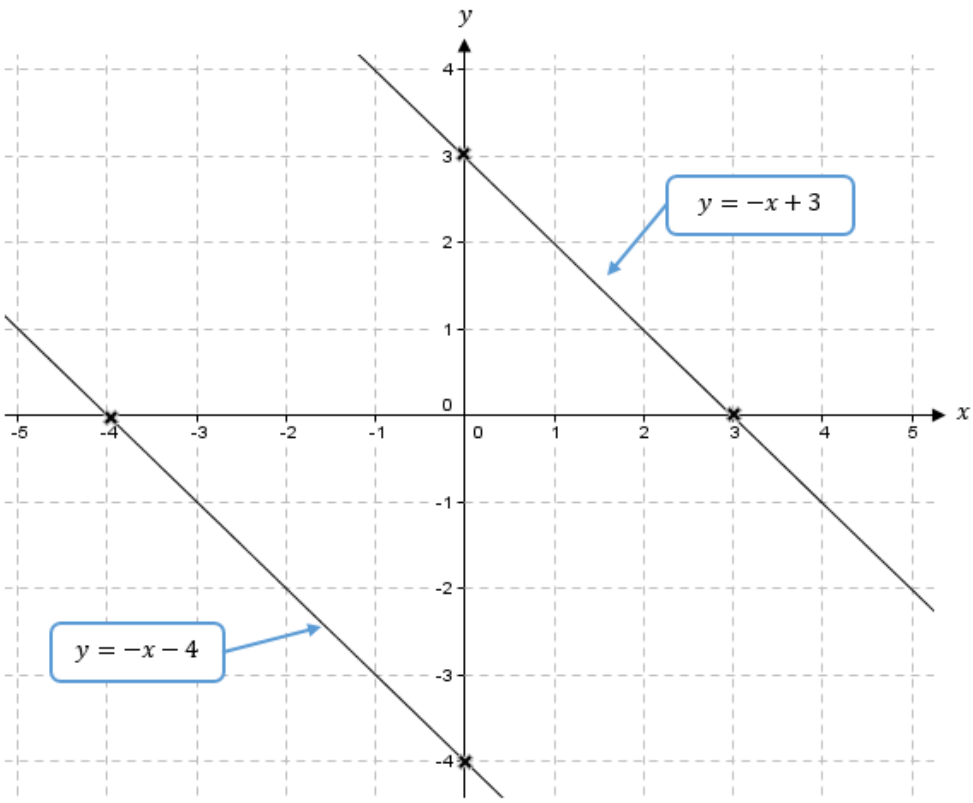


Fig 5.10

We can obtain gradient of a line either from

a) Two given coordinates on the line or

b) The equation of the straight line

a) Gradient from two given coordinates

The definition says gradient is the change of y compared to the change in x. The change in y is the difference between y values and we can denote that by saying $y_1 - y_0$. The change in x is also denoted by saying $x_1 - x_0$. This then will result in the formula for gradient written as

Gradient $= \frac{y_1 - y_0}{x_1 - x_0}$

Let us consider the following example in calculation of gradient using the above formulae given two coordinates on the straight line

Worked Example [7]

Questions

Find the gradient of the line passing through the following coordinates

a) (0; 4) and (–2;0) b) (–2;–4) and (2;8) c) (–1;3) and (1;–1)

Solutions

a) Given (0 ; 4) and (–2 ; 0)

Label your y_1 and y_2 and also your x_1 and x_2 from the two given coordinates

$y_1 = 4,\ y_2 = 0$ and $x_1 = 0,\ x_2 = -2$

<u>Using the formula</u>

Gradient $= \frac{y_1 - y_2}{x_1 - x_2}$,

Gradient $= \frac{4-0}{0-(-2)} = \frac{4}{2} = 2$

b) Given (–2; –4) and (2; 8)

Label your y_1 and y_2 and also your x_1 and x_2 from the two given coordinates

$y_1 = -4, y_2 = 8$ and $x_1 = -2, x_2 = 2$

<u>Using the formula</u>

Gradient $= \frac{y_1 - y_2}{x_1 - x_2}$,

Gradient $= \frac{-4-8}{-2-2} = \frac{-12}{-4} = 3$

c) Given (–1;3) and (1; –1)

Label your y_1 and y_2 and also your x_1 and x_2 from the two given coordinates

$y_1 = 3, y_2 = -1$ and $x_1 = -1, x_2 = 1$

Using the formula

Gradient $= \frac{y_1 - y_0}{x_1 - x_0}$,

Gradient= $\frac{3-(-1)}{-1-1} = \frac{4}{-2} = -2$

Do this yourself;

Find the gradient of the line passing through (–3; 1) and (4; 3) in the space below

If this is done correctly, your gradient should be $\frac{2}{7}$

Now let's look at how we can find the gradient from the equation of a straight line.

5.3.3 Finding the gradient from the equation of a straight line

If given a straight line in the form $y = mx + c$, m represents the gradient of the line and c represents the y intercept.

If given the straight line in the form $ax + by = c$, to find the gradient, make y the subject of the formulae.

Consider the following example on how gradient is obtained from a straight line

Worked Example [8]

Questions

Find the gradient of the straight line represented by the following equations

a) $y=-2x+4$

b) $2y=4x-10$

c) $2x+3y=6$

Solutions

a) The equation of the line is $y = -2x + 4$, here y is already the subject of the formulae. Therefore the gradient is the coefficient of x in this case it is –2 and our gradient is –2

b) The equation of the line is $2y = 4x - 10$. To find the gradient of the line we have to make y the subject of the formulae

$2y = 4x - 10$ (divide both sides of every term by 2 to make y the subject of the formula)

$\frac{2y}{2} = \frac{4x}{2} - \frac{10}{2}$

$y = 2x - 5$

The gradient = 2 (the coefficient of x)

c) The equation of the line is $2x + 3y = 6$, you have to make x the subject of the formulae

$3y = -2x + 6$

Use this space below to find the gradient

If you have done everything correctly, your gradient should be $-\frac{2}{3}$

Activity (*5.3*) Gradient of a Straight Line

Questions

1) Find the gradient of the line passing through the following points

(a) (5; 4) and (0; 0)

(b) (6; –5) and (2; 3)

(c) (–3;1) and (4;3)

2) Find the gradient of the lines with the following equations

(a) $-3y=6x-12$ (b) $2y=3x+4$

(c) $3x+4y=8$ (d) $5x-2y=-6$

Answers

1) (a) $\frac{4}{5}$ (b) –2 (c) $\frac{2}{7}$

2) (a) –2 (b) 1.5 (c) –0.75 (d) 2.5

5.4 CUBIC FUNCTION/GRAPH

In a Cubic function, the highest power of the variable 3. Here are examples of cubic functions

$y = x^3$ $y = 2x^3 + x^2$ $y = 4x - x^3$

The cubic function has two turning points

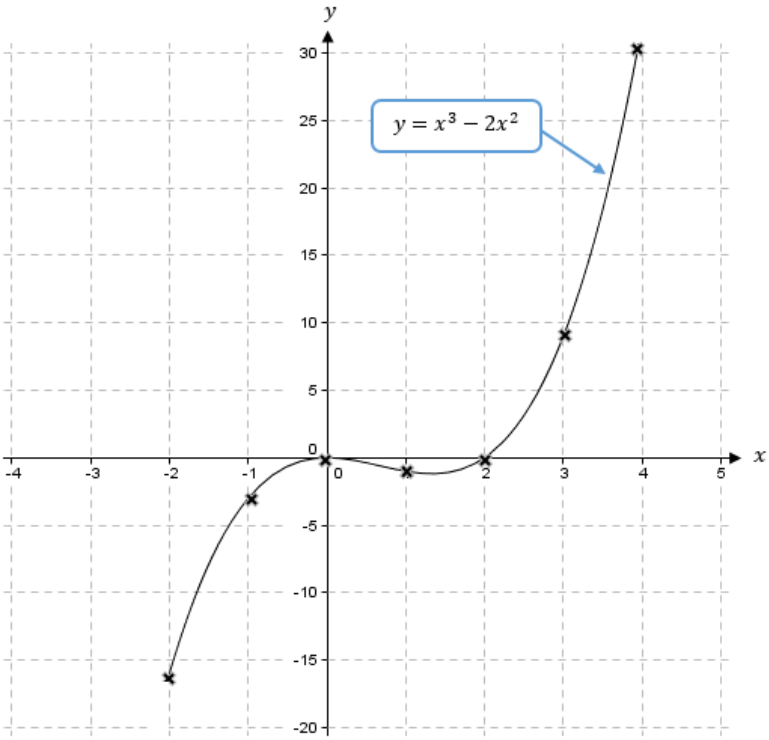


Fig 5.10-2

5.4.1 Plotting the Cubic function

These are the steps followed when plotting a Cubic function:

- Step 1 – Draw up a table of values
- Step 2 – Plot coordinates
- Step 3 – Join the coordinates with a smooth curve

Let's consider the following example on how to plot the Cubic function.

Worked Example [9]

Questions

a) Complete the following table of values for $y = x^3 + 2$

x	–3	–2	–1	0	1	2	3
y			1	2	3		

b) Using a scale of 2cm to represent the x-axis and 2 cm to represent 5 units, draw the graph of $y = x^3 + 2$

Solutions

a) Find the missing values remember we substitute the x value to get the y value

When $x = -3$;

$y = (-3)^3 + 2$

$y = -25$

When $x = -2$;

$y = (-2)^3 + 2$

$y = -8 + 2$

$y = -6$

Find the remaining values of y when $x = 2$ and when $x = 3$ in the space below

If you working is correct, your table of values will be like the one below

x	–3	–2	–1	0	1	2	3
y	–25	–6	1	2	3	10	29

Mark the coordinates on the Cartesian plane and join with a smooth curve to finish off the graph.

Do this yourself

The following is a table of values for $y = x^3$

x	–3	–2	–1.5	–1	0	1	1.5	2	3
y	–27	–8	–3.4	–1	0	1	3.4	8	27

Mark the coordinates and plot the graph of $y = x^3$

5.4.2 Area under the curve

The area enclosed by the curve and other given lines is calculated using either the trapezium or counting squares method.

Let us consider the following example on how to calculate the area under a Cubic function

Worked Example [10]

Questions

The following is the graph of $y = x^3 + 4x + 2$

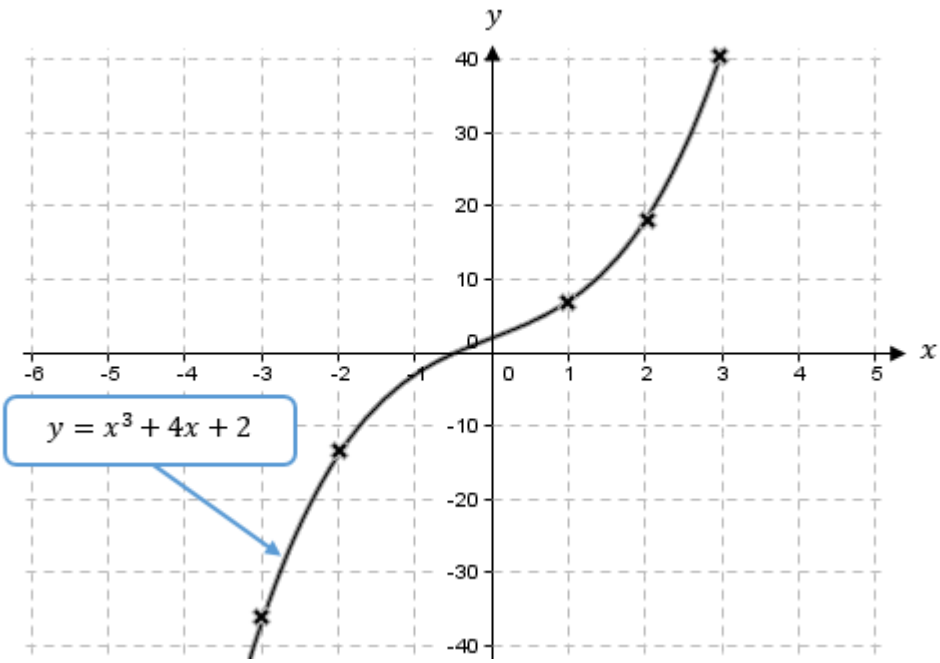


Fig 5.11

On the graph, calculate the area bounded by the curve, $x - \text{axis}$, $x = 1$ and $x = 3$

Solution

Using the graph above, show by drawing $x = 1$ and $x = 3$ to show the area to be calculated as shown below.

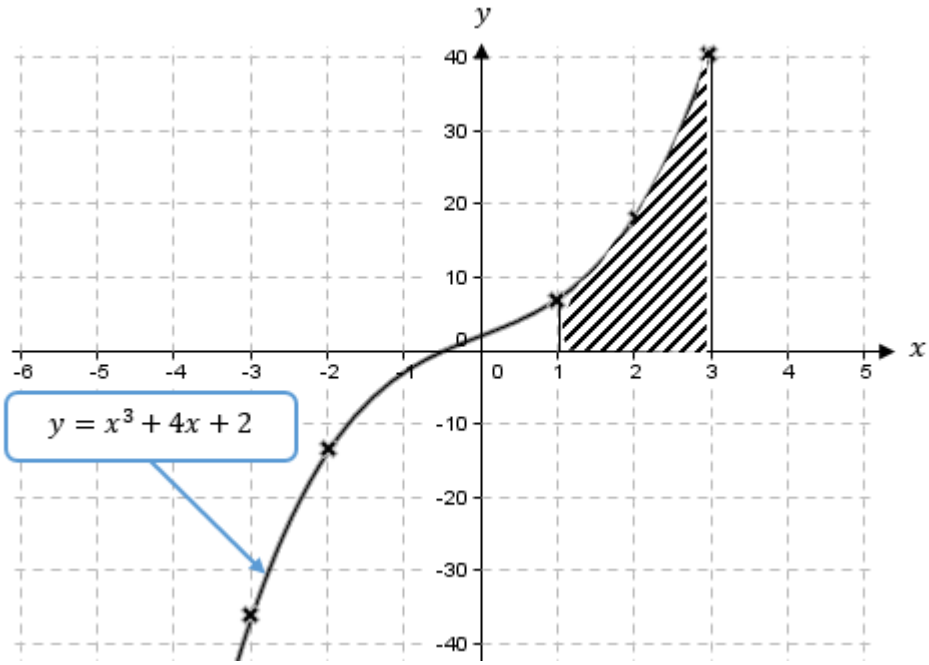


Fig 5.12

Counting squares method	Trapezium method
$0.1 \times 1 \times 450 = 45$ units²	A gives $\frac{1}{2}(7 + 22) = 14.5$ units² B gives $\frac{1}{2}(22 + 41) = 31.5$ units² Area $= A + B = 46$ units²

Do this yourself

On the graph below is $y = x^3 - 2x^2$. Calculate the area enclosed by the curve, $x -$ axis, $x = 2$ and $x = 3$

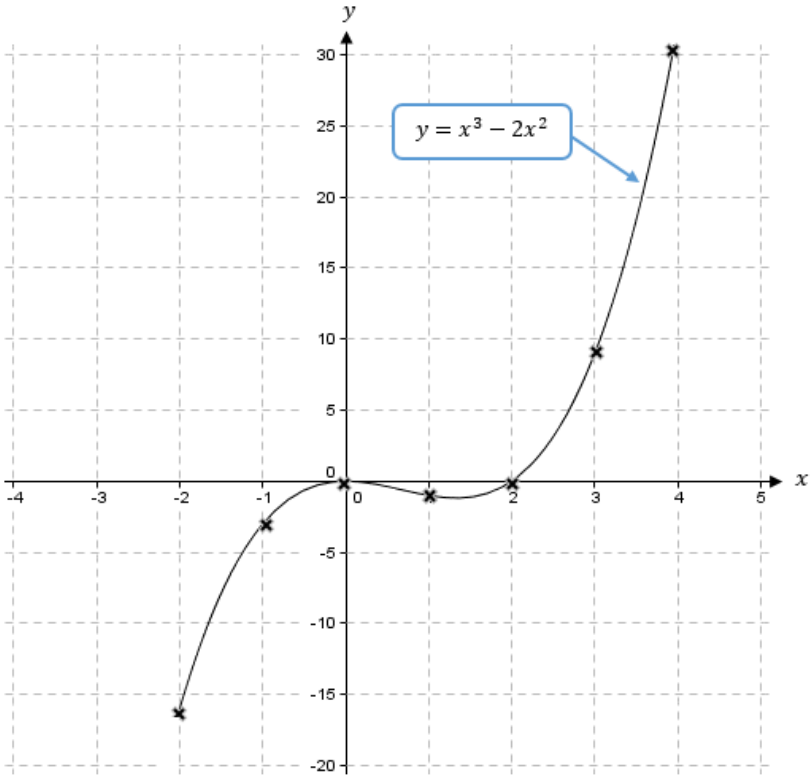


Fig 5.13

Do your working here

Counting squares method	Trapezium method

5.4.3 Gradient on a Cubic Function

These are the steps followed when calculating the gradient on a Cubic function.

Step 1 – Draw a tangent on the point in question

Step 2 – We calculate our gradient using either one of the two methods

a) Using two coordinates on the tangent line

b) Drawing a right angled triangle

Consider the following example, on how to calculate the gradient on a Cubic function

Worked Example [11]

Question

On the graph of $y = x^3 - 2x^2$ below

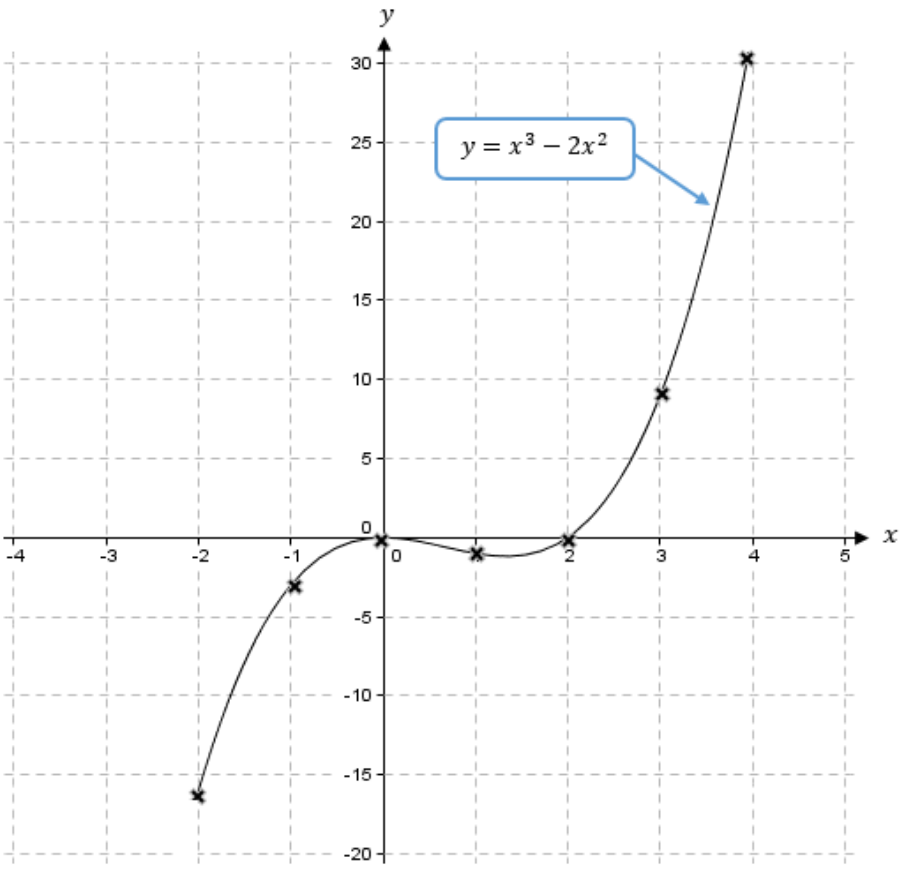


Fig 5.14

Calculate the gradient at the point $x = -1$

Solution

On the graph of $y = x^3 - 2x^2$ above, draw a tangent passing through the point $x = -1$ as shown below

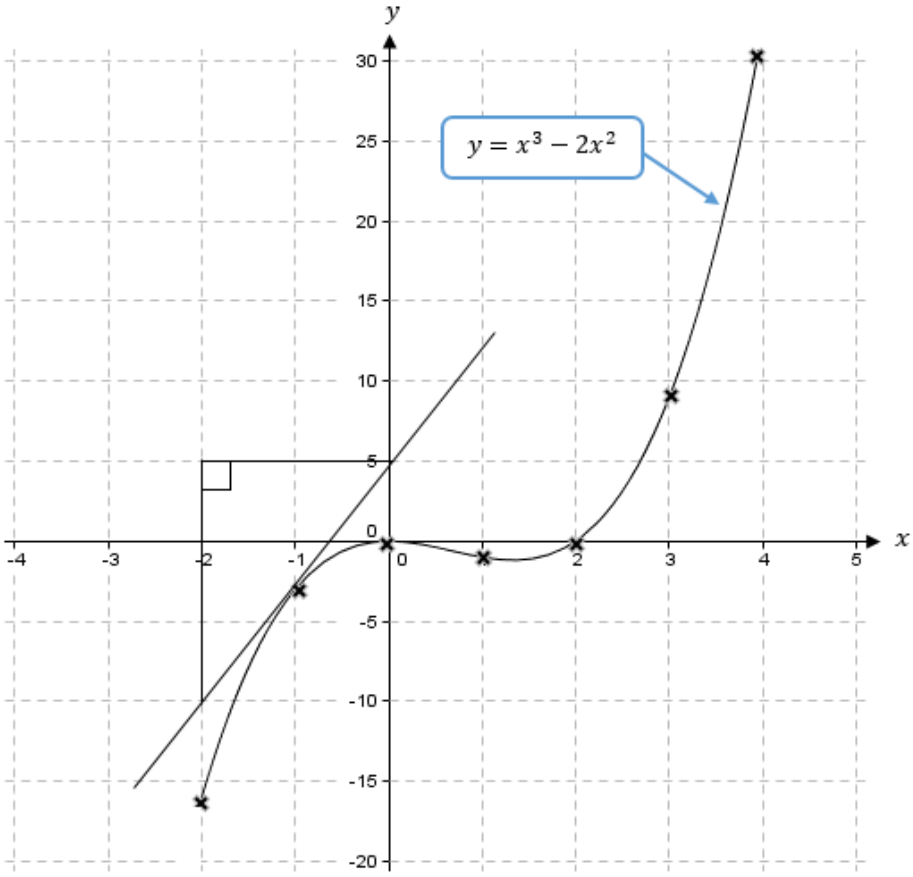


Fig 5.15

Now let us calculate the gradient

Using two coordinates on the tangent line	Using a right angled triangle
The coordinates are (0 ; 5) and (–2 ; –10) Gradient = $\frac{5-(-10)}{0-(-2)} = \frac{15}{2} = 7{,}5$	Gradient = $\frac{15}{2} = 7{,}5$

Do this yourself

The following is the graph of $y = x^3$

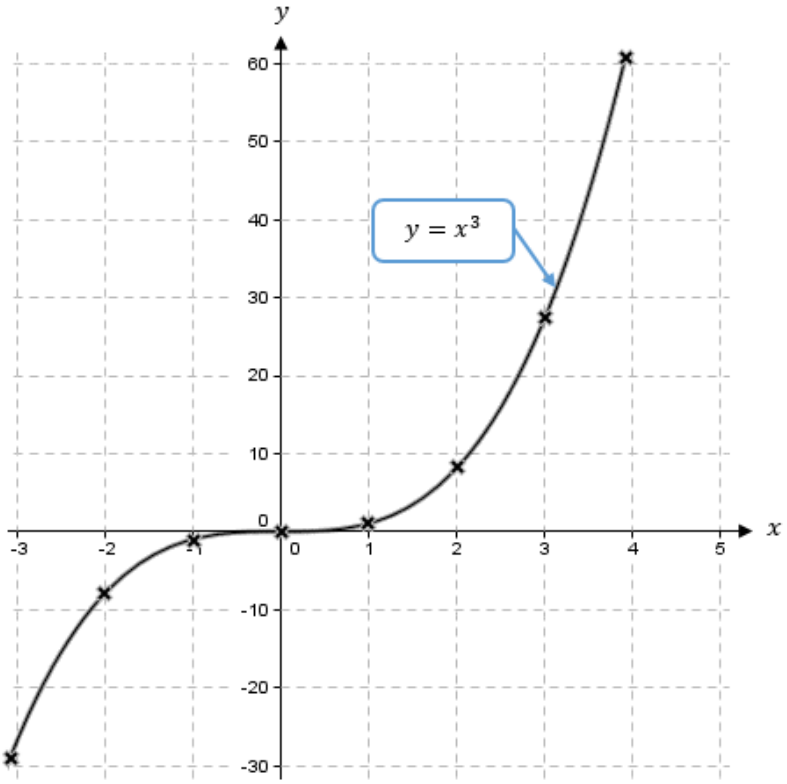


Fig 5.16

Draw a tangent at $x = 2$ and calculate the gradient

Using two coordinates on the tangent line	Using a right angled triangle

How do you know that your computations for area are correct above? Yes! The answers from the two computations should be approximately equal.

5.4.4 Roots of a cubic function

In the Cubic function, usually when a straight line intersects a Cubic function we get three roots. Let us look at how we can find the roots on a cubic function in the following example.

Worked Example [12]

Question

The following is the graph of $y = x^3 + 4x + 2$

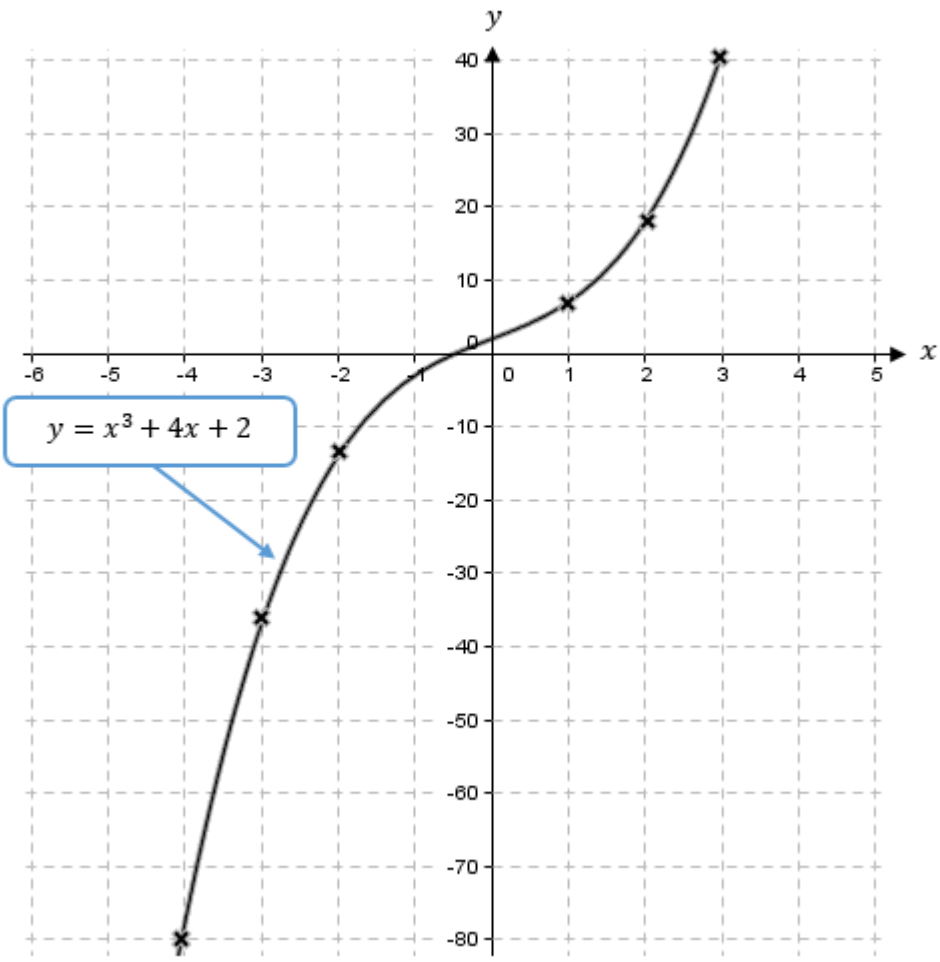


Fig 5.17

Using the graph, find the roots of the equation $0 = x^3 + 4x + 2$

Solution

We subtract the equation ($0 = x^3 + 4x + 2$) from the original function ($y = x^3 + 4x + 2$)

$$\begin{array}{rl} & y = x^3 + 4x + 2 \\ - & 0 = x^3 + 4x + 2 \\ \hline & y - 0 = x3 - x3 + 4x - 4x + 2 - 2 \text{ which gives } y = 0 \\ \hline \end{array}$$

Now you draw the line $y = 0$ on the same axis above as shown below

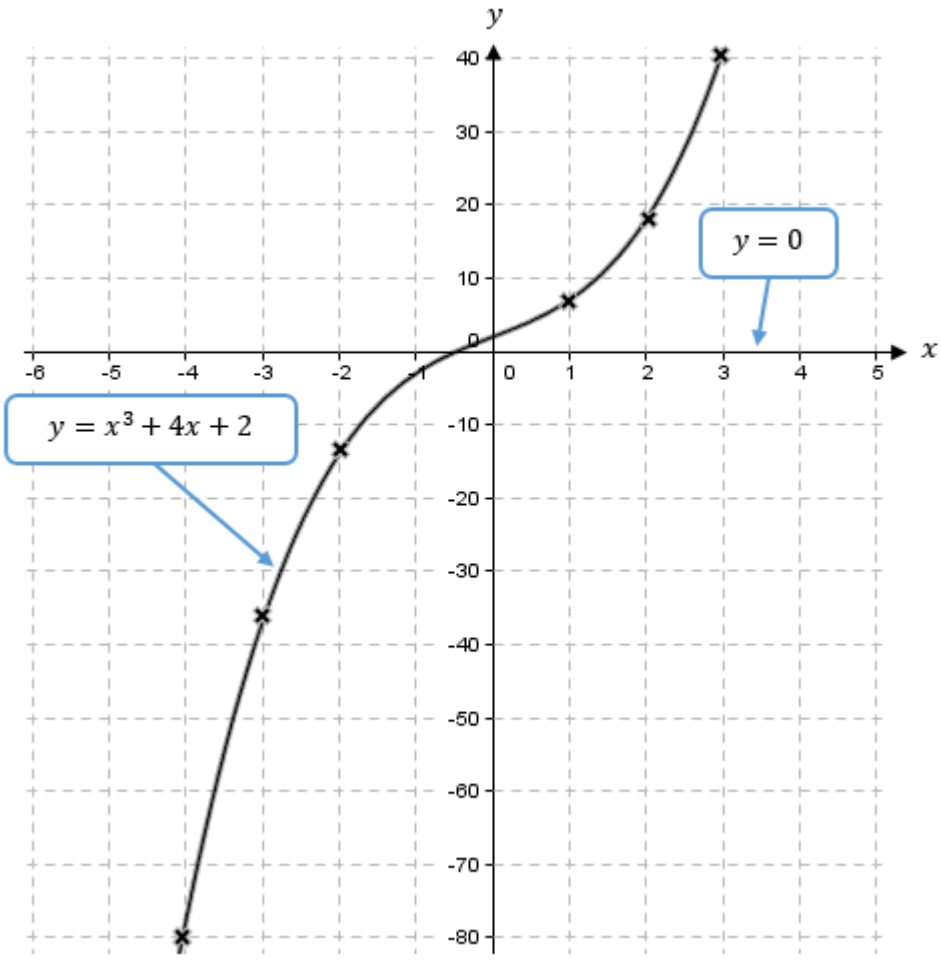


Fig 5.18

The line will intersect the curve at a point where the root is –0,6

Do this yourself

Draw the graph of $y = x^3 - 3x$

Using your graph

(a) Find the roots of the equation $x^3 - 3x = 0$ in the space below

(b) Solve the equation $x^3 - 3x = 2x + 2$ in the space below

5.4.5 Turning points of the Cubic function

A Cubic function has two turning points, that is, a maximum and a minimum (and a saddle which is beyond the scope of this level)

Let us consider the following example and find the turning points of a Cubic function

Worked Example [13]

Question

From the graph of $y = 3x - x^3$ below, write down the coordinates of the turning point.

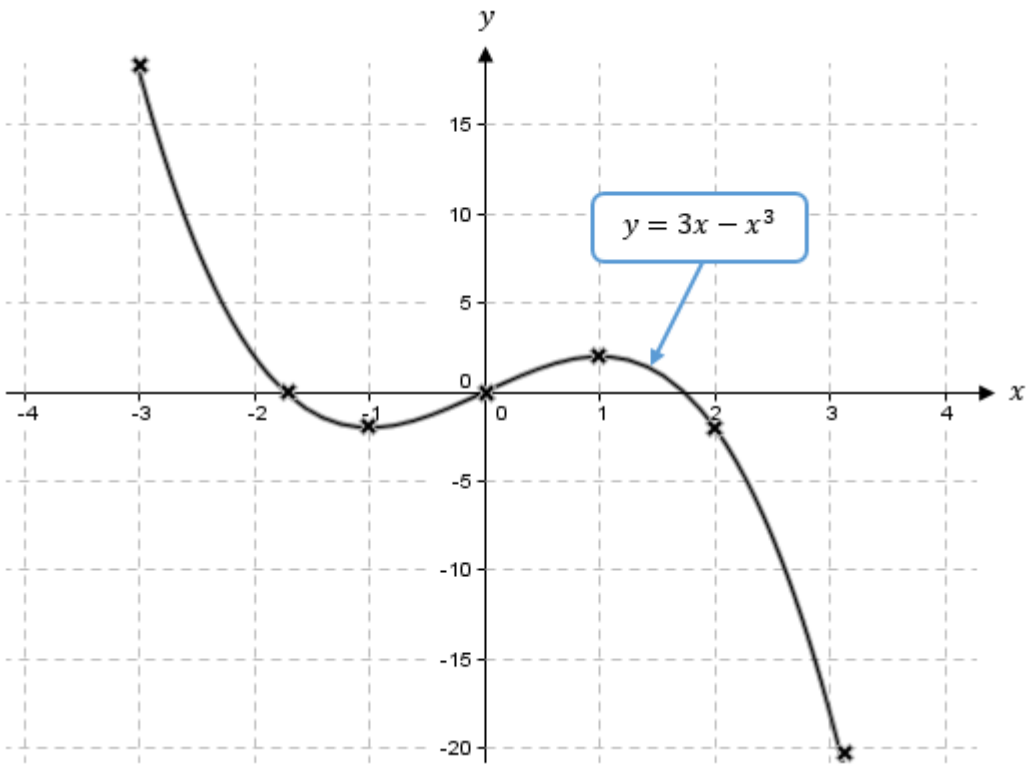


Fig 5.20

Solution

There are two turning points;

Maximum turning point coordinates (1; 2)

Minimum turning point coordinates (–1; –2)

Do this yourself

The following is the graph of $y = x^3 - 12x$

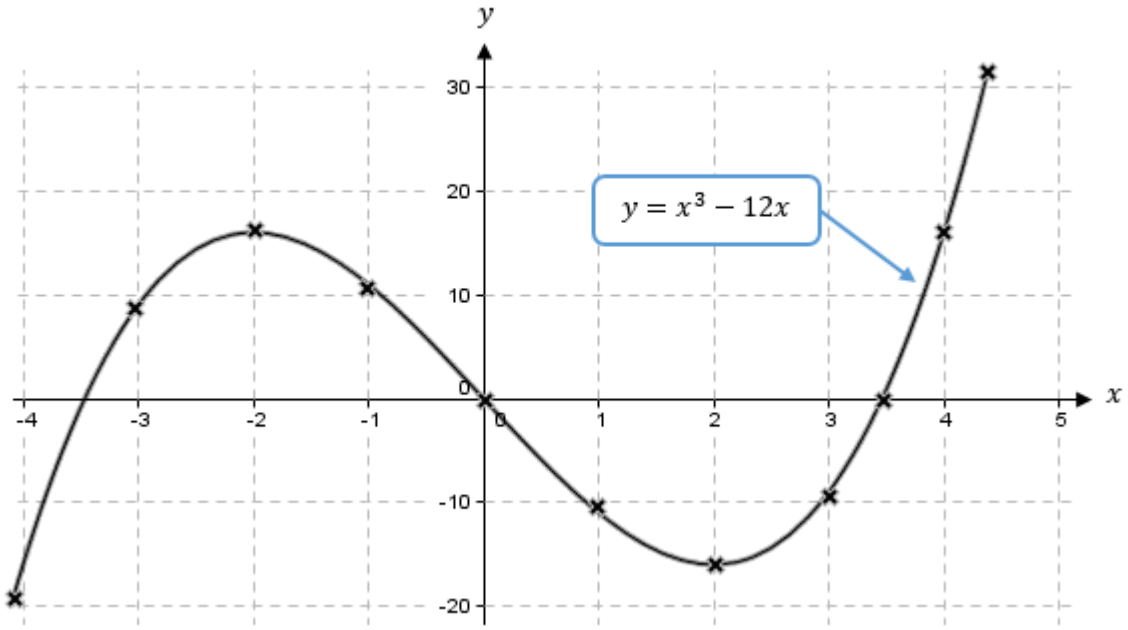


Fig 5.21

Using your graph, state the maximum and minimum values of y.

Maximum y= .....

Minimum y= .....

Activity (5.4) Cubic Functions

Questions

1) Below is an incomplete table of values for $y = x - x^3$

x	–3	–2	–1,5	0	1,5	2	3
y	a	6	1.9	0	b	–6	–24

(a) Find the value of a and the value of b

(b) Using a scale of 2cm to represent 1 unit on the x-axis and 2 cm to represent 5 units on the y-axis, draw the graph of $y = x - x^3$

(c) Use you graph to find

i. Find the gradient of the curve at the point $x = ?$

ii. Write down the coordinates of the maximum turning point

iii. Find the roots of the equation $x - x^3 = 0$

iv. Solve the equation $2 - 2x = x - x^3$

(d) Estimate , using your graph the area enclosed by the curve, the $x -$ axis, the lines $x = 1$ and $x = 3$

2) The table below shows an incomplete table of values for $y = x^3 + 2x + 3$

x	–4	–3	–2	–1	0	1	2	3	4
y		m					n		

(a) Find the values of m and n

(b) Using a 2cm to represent 1 unit on the x-axis and 2cm to represent 5 units on the y–axis, draw the graph of $y = x^3 + 2x + 3$

(c) Using your graph

I. Find the gradient of the curve at $x = 2$

II. Find the roots of the equation $x^3 + 2x + 3 = 0$

III. State the values of y at the turning points

(d) Calculate an estimate of the area enclosed by the curve, the x-axis, $x = -2$ and $x = -3$

(e) Show on the graph the region for which we will calculate the area.

Answers

1) (a) a = 24 b = –1,9

(b)

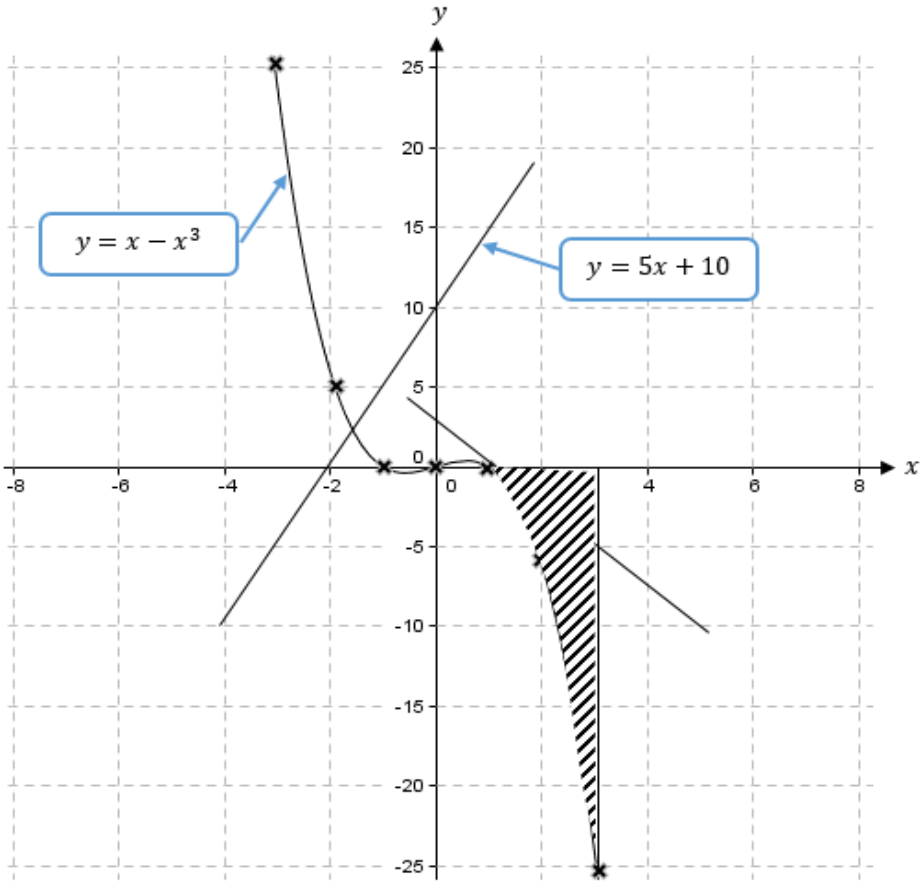


Fig 5.22

(c) i) Gradient = –3,75

ii) no maximum or minimum …(it's a saddle which is beyond the scope of this book)

iii) $x = 0$

iv) –1.55

(d) 15 units²

2) (a) m = –28 n = 15

(b)

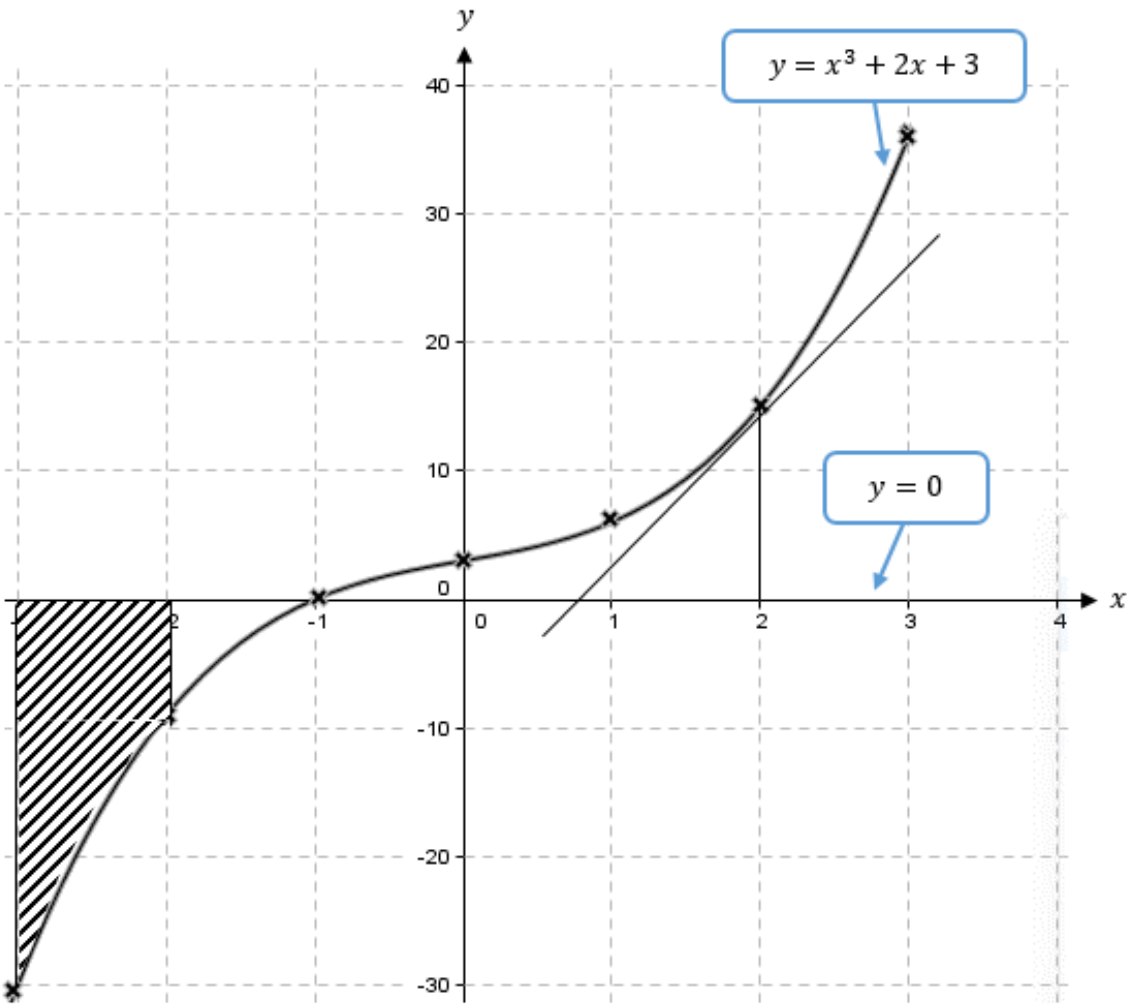


Fig 5.23

(c) i) Gradient = 12.5

ii) $x = -1$

iii) no turning point

(d) 18,5 units²

SUMMARY

In this unit we have about how to interpret the functional notation, construct a table of values from a given function, use given scale correctly to construct the functions, draw graphs (functions) of the form

- $y = mx + c$ (linear function)
- $y = x^3$ (cubic function)

We also have learnt how to estimate the area under the curve, draw tangents at given points, estimate the gradient of lines and curves at given points, solve equations graphically and use graphs to determine the maximum and minimum turning points.

FURTHER READING

1. Macrae, M, Madungwe, L and Mutangadura, A (2017) New General Mathematics Book 3, Pearson Capetown.
2. Macral, M, Madungwe, L and Mutangadura, A (2017) New General Mathematics Book 4, Pearson Capetown.
3. Meyers, C; Graham, B; Dawe, L (2004) Mathscape Working Mathematically, 9th Ed, MacMillan
4. Pimentel R and Wall T (2011) International Mathematics, Hodder UK
5. Rayner D (2005) Extended Mathematics. Oxford New York

UNIT 6 – MATRICES

CONTENTS

6.1 INTRODUCTION

In this unit you are going to cover mathematical operations in singular and non-singular matrices. You are also going to use matrices as operators, that is, solving matrices using simultaneous equations.

OBJECTIVES

After going through this unit, you should be able to:

- perform operations with matrices
- solve problems involving 2×2 singular and non-singular matrices
- find the determinant of a2×2matrix
- find the inverse of 2×2 non-singular matrix
- solve simultaneous problems using the matrix method

KEY TERMS

Matrix: – is the rectangular arrangement of elements or objects.

Order of a matrix: – is the number of rows and columns of a given matrix.

Determinant: - determines whether the matrix is invertible or not.

Singular matrix: - is a matrix which has a determinant of zero, it has no inverse.

TIME: you should not spend more than 10 hours on this unit.

STUDY SKILLS

The key skill to mastery of mathematical concepts is practice. You need to solve as many problems on the topic on Matrices as possible for you to grasp concepts in this topic.

6.2 ORDER OF A MATRIX

Order of a matrix is defined by stating number of rows (horizontally arranged) and number of columns (vertically arranged).There are 3 rows and 2 columns, hence it's a 3×2 matrix

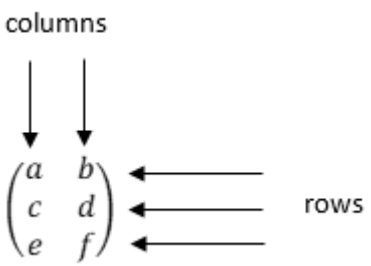


Fig 6.1 – Order of matrix

Consider the following examples on how to find the order of matrices.

Worked Example [1]

Questions

State the order of the following matrices.

a) $A = \begin{pmatrix} 2 & 1 \\ 2 & 3 \\ 4 & 1 \end{pmatrix}$

b) $B = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$

c) $C = \begin{pmatrix} 5 & 3 & 4 \\ 3 & 1 & 2 \\ 5 & 3 & 2 \end{pmatrix}$

d) $D = \begin{pmatrix} 1 & 3 & -7 \end{pmatrix}$

Solutions

a) For you to state the order of a matrix A you have to count number of rows in the matrix first and then number of columns. Did you see that there are 3 rows and 2 columns? We call this a 3 by 2 matrix because it has 3 rows and 2 columns. 3 x 2

b) Matrix B is a 2 by 1 column matrix. It is called a column matrix. It is called a column matrix because it has only one column. 2 x 1

c) Matrix C has a special name given to it. It is called a square matrix because it has equal number of rows and columns. It is a 3 by 3 square matrix. It can also be referred to as 'a square matrix of order 3'.

d) D is a matrix which has 1 row and 3 columns. When you see a matrix with only one row remember that it has a special name. It is called a row matrix.

Activity (*6.1*) Order of a matrix

Questions

1) State the order of the following matrices.

a) $\begin{pmatrix} 0 & 2 & 5 \\ 1 & 3 & 4 \end{pmatrix}$

b) $\begin{pmatrix} x & -4 \\ 3 & 6 \\ 6 & y \\ 8 & 5 \end{pmatrix}$

c) $\begin{pmatrix} 4 & 5 & -9 & 0 \\ 3 & 4 & 5 & 7 \\ 5 & -2 & 4 & 1 \end{pmatrix}$

d) $\begin{pmatrix} 3 & 2 \\ 1 & 3 \end{pmatrix}$

2) State the special name given to the following matrices

a) $\begin{pmatrix} 3 & 3 & 7 \\ 2 & 6 & 9 \\ 5 & -8 & 0 \end{pmatrix}$

b) $\begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$

c) $\begin{pmatrix} 1 & -8 & 6 & 3 \end{pmatrix}$

Answers

1 a) 2×3

b) 4 × 2

c) 3 × 4

d) 2 × 2

2 a) Square matrix of order 3

b) 3 by 1 column matrix

c) 1 by 4 row matrix

6.3 ADDITION AND SUBTRACTION OF MATRICES

When you add or subtract matrices, you add or subtract corresponding elements of the given matrices.

Note that: you can only add or subtract matrices which have the same order, if matrices are not of the same order it is impossible to add or subtract them.

Consider the following examples below.

Worked Example [2]

Questions

Given that A= $\begin{pmatrix} 1 & 3 \\ -3 & 4 \end{pmatrix}$ B= $\begin{pmatrix} 5 & 4 \\ 7 & 2 \end{pmatrix}$ C= $\begin{pmatrix} -4 & 8 \\ 6 & 9 \end{pmatrix}$ D= $\begin{pmatrix} 5 & 3 & 1 \\ 5 & 2 & 7 \end{pmatrix}$

a) $A + B$

b) $B - C$

c) $A + C - B$

d) $A + D$

Solutions

a) Since matrix A and matrix B are matrices of the same order that is (2 by 2) you can proceed to add them.

So, A+ B= $\begin{pmatrix} 1 & 3 \\ -3 & 4 \end{pmatrix} + \begin{pmatrix} 5 & 4 \\ 7 & 2 \end{pmatrix}$

$= \begin{pmatrix} 1+5 & 3+4 \\ -3+7 & 4+2 \end{pmatrix}$,

$= \begin{pmatrix} 6 & 7 \\ 4 & 6 \end{pmatrix}$, in this example youcan see that elements (1 and 5), (3 and 4), (-3 and 7) and elements (4 and 2) are corresponding values that is why it was possible to add them.

b) Now repeat the same method you have applied above, but mind that you are now subtracting the corresponding values.

$B - C = \begin{pmatrix} 5 & 4 \\ 7 & 2 \end{pmatrix} - \begin{pmatrix} -4 & 8 \\ 6 & 9 \end{pmatrix}$

$= \begin{pmatrix} 5 - 4 & 4-8 \\ 7-6 & 2-9 \end{pmatrix}$

$= \begin{pmatrix} 9 & -4 \\ 1 & -7 \end{pmatrix}$

c) $A + C - B = \begin{pmatrix} 1 & 3 \\ -3 & 4 \end{pmatrix} + \begin{pmatrix} -4 & 8 \\ 6 & 9 \end{pmatrix} - \begin{pmatrix} 5 & 4 \\ 7 & 2 \end{pmatrix}$

$= \begin{pmatrix} 1-4-5 & 3+8-4 \\ -3+6-7 & 4+9-2 \end{pmatrix}$

$= \begin{pmatrix} -8 & 7 \\ -4 & 11 \end{pmatrix}$

d) If you look at the order of matrix A and matrix D, you can see that matrix A is a 2 by 2 matrix whilst matrix D is a 2 by 3 matrix. Since the 2 matrices are not of the same order you cannot proceed with addition. It is impossible to add.

Referring to the examples above, now attempt the following activity.

Activity (*6.2*) Addition and subtraction

Questions

a) $\begin{pmatrix} -1 & -7 \\ 3 & 8 \end{pmatrix} + \begin{pmatrix} 2 & 8 \\ 3 & -6 \end{pmatrix}$

b) $\begin{pmatrix} 2 & 6 \\ 3 & 8 \\ -1 & 5 \end{pmatrix} - \begin{pmatrix} -7 & 4 \\ 6 & 0 \\ -6 & -3 \end{pmatrix}$

c) $\begin{pmatrix} 1 & -6 \\ -2 & 4 \end{pmatrix} + \begin{pmatrix} 0 \\ 2 \end{pmatrix}$

d) $\begin{pmatrix} 4 & -8 \\ 5 & 4 \end{pmatrix} - \begin{pmatrix} -7 & -6 \\ 0 & 4 \end{pmatrix} + \begin{pmatrix} 3 & -3 \\ 6 & -5 \end{pmatrix}$

e) $\begin{pmatrix} 0 & 6 \end{pmatrix} + \begin{pmatrix} 4 \\ -5 \end{pmatrix}$

Answers

1) a) $\begin{pmatrix} 1 & 1 \\ 6 & 2 \end{pmatrix}$

b) $\begin{pmatrix} -9 & 2 \\ -3 & 8 \\ 5 & 8 \end{pmatrix}$

c) Impossible

d) $\begin{pmatrix} 14 & -5 \\ 11 & -5 \end{pmatrix}$

e) Impossible

6.4 SCALAR MULTIPLICATION AND EQUAL MATRICES

A scalar is a numerical matrix multiplier, which have an effect of multiplying every element in the given matrix for example $3\begin{pmatrix} x \\ y \end{pmatrix}$ matrix. Here the scalar is 3

You must be aware that a scalar can be either positive or negative and can also be a fraction.

Worked Example [3]

Questions

Given that X = $\begin{pmatrix} 3 & 2 \\ 4 & -5 \end{pmatrix}$, Y = $\begin{pmatrix} 4 \\ 2 \end{pmatrix}$ and Z = $\begin{pmatrix} 3 & 3 & 2 \end{pmatrix}$. Find

a) 3X

b) $\frac{1}{2}$Y

c) –5Z

Solutions

a) $3X = 3\begin{pmatrix} 3 & 2 \\ 4 & -5 \end{pmatrix}$

b) $\frac{1}{2}Y = \frac{1}{2}\begin{pmatrix} 4 \\ 2 \end{pmatrix}$

c) $-5Z = -5(3 \quad 3 \quad 2)$

$= (-15 \quad -15 \quad -10)$

$= \begin{pmatrix} 9 & 6 \\ 12 & -15 \end{pmatrix}$

$= \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

Worked Example [4]

Questions

In this example we are going to make use of scalar multiplication in solving matrices

Given that $X = \begin{pmatrix} 3 & -3 \\ 4 & 5 \end{pmatrix}$, $Y = \begin{pmatrix} x & -6 \\ 8 & 10 \end{pmatrix}$ and $Z = \begin{pmatrix} 1 & 8 \\ 7 & 6 \end{pmatrix}$. Find

a) $3X + 2Z$

b) x given that $X = \frac{1}{2}Y$

c) Find the value of a and b given that $\begin{pmatrix} a & 1 \\ -1 & 2 \end{pmatrix} - \begin{pmatrix} -4 & 5 \\ 1 & 6 \end{pmatrix} = \begin{pmatrix} 6 & -4 \\ -2 & b \end{pmatrix}$

Solution

a) Multiply matrix X by 3 and matrix Z by 2 and then proceed with your addition

$3X + 2Z$

$= 3\begin{pmatrix} 3 & -3 \\ 4 & 5 \end{pmatrix} + 2\begin{pmatrix} 1 & 8 \\ 7 & 6 \end{pmatrix}$

$= \begin{pmatrix} 9 & -9 \\ 12 & 15 \end{pmatrix} + \begin{pmatrix} 2 & 16 \\ 14 & 12 \end{pmatrix}$

$= \begin{pmatrix} 11 & 7 \\ 26 & 27 \end{pmatrix}$

b) $X = \frac{1}{2}Y$, multiply matrix Y by $\frac{1}{2}$ and proceed with your workings

$\begin{pmatrix} 3 & -3 \\ 4 & 5 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} x & -6 \\ 8 & 10 \end{pmatrix}$, equate corresponding values

$= \frac{x}{2}$

solving for x we get

$x = 6$

c) $\begin{pmatrix} a & 1 \\ -1 & 2 \end{pmatrix} - \begin{pmatrix} -4 & 5 \\ 1 & 6 \end{pmatrix} = \begin{pmatrix} 6 & -4 \\ -2 & b \end{pmatrix}$

by subtracting corresponding values we get;

$\begin{pmatrix} a - -4 & 1 - 5 \\ -1 - 1 & 2 - 6 \end{pmatrix} = \begin{pmatrix} 6 & -4 \\ -2 & b \end{pmatrix}$

simplifying we get;

$\begin{pmatrix} a + 4 & -4 \\ -2 & -4 \end{pmatrix} = \begin{pmatrix} 6 & -4 \\ -2 & b \end{pmatrix}$

by equating corresponding values we get;

$a + 4 = 6,$

$a = 2$

The value of b is

$b = -4$

Worked Example [5]

Question

Find a 2×2 matrix M such that $3M + \begin{pmatrix} -3 & 0 \\ -2 & 5 \end{pmatrix} = \begin{pmatrix} 6 & 3 \\ 4 & 2 \end{pmatrix}$

Solution

We first regroup the matrices and we get;

$3M = \begin{pmatrix} 6 & 3 \\ 4 & 2 \end{pmatrix} - \begin{pmatrix} -3 & 0 \\ -2 & 5 \end{pmatrix}$

$3M = \begin{pmatrix} 9 & 3 \\ 6 & -3 \end{pmatrix}$,

since matrix M is multiplied by 3, then we divide the matrix by 3

$\frac{3}{3} M = \frac{1}{3}\begin{pmatrix} 9 & 3 \\ 6 & -3 \end{pmatrix}$

$\therefore\ M = \begin{pmatrix} 3 & 1 \\ 2 & -1 \end{pmatrix}$

Now that you have looked at these examples, attempt the following activity

Activity (*6.3*) Scalar Multiplication

Questions

1) Solve the following equations, in each case X represents a matrix.

a) $2X + \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix} = \begin{pmatrix} 8 & 7 \\ 3 & 10 \end{pmatrix}$

b) $\frac{1}{3}X = \begin{pmatrix} 2 & 3 \\ 1 & 0 \end{pmatrix}$

c) $4X = \begin{pmatrix} 12 & 4 & 8 \\ 0 & 16 & 12 \end{pmatrix}$

2) Given that $\begin{pmatrix} x & 4 & 2 \\ 3 & 0 & 1 \end{pmatrix} + \begin{pmatrix} -4 & 7 & -5 \\ 3 & w & -3 \end{pmatrix} = \begin{pmatrix} 2 & y & -3 \\ 6 & -4 & -2 \end{pmatrix}$, find w, x and y

Answers

1) a) $X = \begin{pmatrix} 3 & 2 \\ 1 & 3 \end{pmatrix}$ b) $X = \begin{pmatrix} 6 & 9 \\ 3 & 0 \end{pmatrix}$ c) $X = \begin{pmatrix} 3 & 1 & 2 \\ 0 & 4 & 3 \end{pmatrix}$

2) $w = -4,\ x = 6 \quad y = 11$

6.5 MULTIPLICATION

Matrices can be multiplied if and only if number of columns in the first matrix is the same as number of rows in the second matrix. Let A be a matrix with order $\mathbf{p} \times \mathbf{r}$ and let matrix B be a matrix of order $\mathbf{r} \times \mathbf{m}$, then

(i) **AB =** $\mathbf{p} \times \mathbf{r} \times \mathbf{r} \times \mathbf{m}$ - since the number of columns is the same as the number of rows then multiplication is possible and the order of the resultant matrix is $\mathbf{p} \times \mathbf{m}$

(ii) **BA =** $\mathbf{r} \times \mathbf{m} \times \mathbf{p} \times \mathbf{r}$ - impossible. Why? Because the number of columns in the first matrix is not equal or the same as the number of rows in the second matrix.

So, AB≠ BA. Consider the following examples.

Worked Example [6]

Questions

Find the product of the following matrices if possible

a) $\begin{pmatrix} 2 & 3 \end{pmatrix}\begin{pmatrix} -1 \\ 4 \end{pmatrix}$

b) $\begin{pmatrix} 4 \\ 2 \end{pmatrix}\begin{pmatrix} 1 \\ 3 \end{pmatrix}$

c) $\begin{pmatrix} 1 & 5 \\ 2 & 8 \end{pmatrix}\begin{pmatrix} -2 \\ 3 \end{pmatrix}$

Solution

a) From the question, the first matrix is a 1× 2 and the second matrix is a 2×1, if you want to find the product of the 2 matrices it is possible since $1 \times 2 \times 2 \times 1$

$\begin{pmatrix} 2 & 3 \end{pmatrix}\begin{pmatrix} -1 \\ 4 \end{pmatrix}$=(2× $-1 + 3 \times 4$)

$= (-2 + 12)$

=(10)

b) $\begin{pmatrix} 4 \\ 2 \end{pmatrix}\begin{pmatrix} 1 \\ 3 \end{pmatrix}$, the first matrix is a 2×1 and the second is a 2×1, so it is impossible to multiply the matrices since $2 \times 1 \times 2 \times 1$

c) From the question we first check if the matrices are multipliable, yes they are so we continue with the multiplication

$$\begin{pmatrix} 1 & 5 \\ 2 & 8 \end{pmatrix}\begin{pmatrix} -2 \\ 3 \end{pmatrix} = \begin{pmatrix} (1 \times -2) + (5 \times 3) \\ (2 \times -2) + (8 \times 3) \end{pmatrix}$$

$$= \begin{pmatrix} -2 + 15 \\ -4 + 24 \end{pmatrix}$$

$$= \begin{pmatrix} 13 \\ 20 \end{pmatrix}$$

Worked Example [7]

Questions

Given that A=$\begin{pmatrix} 1 & 3 \\ -1 & 0 \end{pmatrix}$ and $\mathrm{B} = \begin{pmatrix} 4 & 9 \\ -2 & 5 \end{pmatrix}$. Find

a) AB b) BA c) B^2

Solutions

a) AB $= \begin{pmatrix} 1 & 3 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 4 & 9 \\ -2 & 5 \end{pmatrix}$

$$= \begin{pmatrix} (1 \times 4 + 3 \times -2) & (1 \times 9 + 3 \times 5) \\ (-1 \times 4 + 0 \times -2) & (-1 \times 9 + 0 \times 5) \end{pmatrix}$$

$$= \begin{pmatrix} 4 - 6 & 9 + 15 \\ -4 + 0 & -9 + 0 \end{pmatrix}$$

$$= \begin{pmatrix} -2 & 24 \\ -4 & -9 \end{pmatrix}$$

b) BA$= \begin{pmatrix} 4 & 9 \\ -2 & 5 \end{pmatrix}\begin{pmatrix} 1 & 3 \\ -1 & 0 \end{pmatrix}$

$$= \begin{pmatrix} (4 \times 1 + 9 \times -1) & (4 \times 3 + 9 \times 0) \\ (-2 \times 1 + 5 \times -1) & (-2 \times 3 + 5 \times 0) \end{pmatrix}$$

$$= \begin{pmatrix} 4 - 9 & 12 + 0 \\ -2 - 5 & -6 + 0 \end{pmatrix}$$

$$= \begin{pmatrix} -5 & 12 \\ -7 & -6 \end{pmatrix}$$

c) $B^2 = \begin{pmatrix} 4 & 9 \\ -2 & 5 \end{pmatrix}\begin{pmatrix} 4 & 9 \\ -2 & 5 \end{pmatrix}$

$= \begin{pmatrix} (4 \times 4 + 9 \times -2) & (4 \times 9 + 9 \times 5) \\ -2 \times 4 + 5 \times -2) & (-2 \times 9 + 5 \times 5) \end{pmatrix}$

$= \begin{pmatrix} 16 - 18 & 36 + 45 \\ -8 - 10 & -18 + 25 \end{pmatrix}$

$= \begin{pmatrix} -2 & 81 \\ -18 & 7 \end{pmatrix}$

Now that we have gone through a few examples attempt the following activity.

Activity (6.4) Multiplication

Questions

1) Find matrix M such that M + $\begin{pmatrix} 2 & -1 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 5 & 1 \\ 1 & 7 \end{pmatrix}$

2) Given that A= $\begin{pmatrix} 4 & 3 \\ 2 & 1 \end{pmatrix}$, B = $\begin{pmatrix} 0 & -2 \\ 1 & 6 \end{pmatrix}$. Find

   a) AB b) BA c) A^2 d) B^2

3) Find the value of p, q and r

   $\begin{pmatrix} 5 & -1 \\ p & 3 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} q & r \\ 1 & 8 \end{pmatrix}$

Answers

1) M=$\begin{pmatrix} 3 & 2 \\ 1 & 4 \end{pmatrix}$

2) a) AB = $\begin{pmatrix} 3 & 10 \\ 1 & 2 \end{pmatrix}$ b) BA= $\begin{pmatrix} -4 & -2 \\ 16 & 9 \end{pmatrix}$

   c) $A^2 \begin{pmatrix} 22 & 15 \\ 10 & 7 \end{pmatrix}$ d) $B^2 \begin{pmatrix} -2 & -12 \\ 6 & 34 \end{pmatrix}$

3) $q = 11$ $p = 2$ $r = 3$

6.6 DETERMINANT OF 2×2 MATRICES

Given a 2×2 matrix M = $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ the determinant of M= $ad - bc$

Note that the determinant can be denoted by $|M|$

Worked Example [8]

Questions

Given that M=$\begin{pmatrix} 2 & -4 \\ 3 & 6 \end{pmatrix}$, find the determinant of M

Solution

By using the formula $ad - bc$ we substitute the values in the formula so,

Det M= $ad - bc$

=$(2\times 6) - (-4 \times 3)$, note that a is represented by 2, b by -4, c by 3 and d by 6.

=$12-(-12)$

= 24

Worked Example [9]

Questions

a) Given$\begin{pmatrix} 1 & -4 \\ -8 & 4x \end{pmatrix}$ has a determinant of 12, find the value of x

b) Given that A=$\begin{pmatrix} 4 & -2 \\ y & y \end{pmatrix}$ and $|A| = -6$. Find y.

Solutions

a) First we write the formula for calculating determinant and substitute it with the corresponding elements of the matrices;

Det = $ad - bc$, but the determinant is 12 so,

$12 = (1 \times 4x) - (-4 \times -8))$

$12 = 4x - 32$

$44 = 4x$

$\therefore x = 11$

b) $|A|= ad - bc$

$-6=4y + 2y$

$-6 = 6y$

$y = -1$

Now that we have looked at these examples attempt the following activity,

Activity (*6.1*) Determinant

Questions

1) Find the determinant of the following matrices

a) $\begin{pmatrix} 3 & -1 \\ 5 & 4 \end{pmatrix}$ b) $\begin{pmatrix} -2 & -8 \\ 4 & 6 \end{pmatrix}$

2) The determinant of a matrix $\begin{pmatrix} a-1 & 9 \\ 6 & 3 \end{pmatrix}$ is 9. Find the value of a.

3) The matrix $\begin{pmatrix} x+1 & 3 \\ x & 2 \end{pmatrix}$ has a determinant of -2.Find the value of x.

Answers

1 a) 17

b) 20

2) $a = 19$

3) $x = 4$

6.7 INVERSE OF A 2×2 MATRIX

Given a 2×2 matrix M=$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and determinant of M = $ad - bc$

The inverse of M = $\frac{1}{\det M}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

The inverse of matrix M may be written as $\mathbf{M}^{-1}$

Note that; elements a and d have exchanged the positions while b and c are multiplied by -1.

Worked Example [10]

Questions

If M =$\begin{pmatrix} 2 & -6 \\ -1 & 4 \end{pmatrix}$

a) Find the value of the determinant of M

b) Hence write down the inverse of M

Solutions

a) Det M = $\mathrm{ad} - \mathrm{bc}$

$= 2\times 4 - (-6 \times -1)$

$= 2$

b) So, M inverse ($\mathbf{M}^{-1}$) is

$$= \frac{1}{\mathrm{detM}}\begin{pmatrix} \mathrm{d} & -\mathrm{b} \\ -\mathrm{c} & \mathrm{a} \end{pmatrix}$$

$= \frac{1}{2}\begin{pmatrix} 4 & 6 \\ 1 & 2 \end{pmatrix}$, from this example can you see that element 2 and 4 have exchanged positions while -6 and -2 have been multiplied by -1?

Worked Example [11]

Questions

Given the matrix A= $\begin{pmatrix} 3 & 0 \\ 2 & -4 \end{pmatrix}$, find its inverse

Solution

Det A = $\mathrm{ad} - \mathrm{bc}$

$= 3(-4) - 2(0)$

$= -12$

$$\mathrm{A}^{-1} = \frac{1}{\mathrm{detA}}\begin{pmatrix} \mathrm{d} & -\mathrm{b} \\ -\mathrm{c} & \mathrm{a} \end{pmatrix}$$

$$= -\frac{1}{12}\begin{pmatrix} -4 & 0 \\ -2 & 3 \end{pmatrix}$$

Worked Example [12]

Questions

Given that N=$\begin{pmatrix} -1 & 1 \\ 0 & 3 \end{pmatrix}$ find N^{-1}

Solution

$|N| = ad - bc$

$= -1(3) - 1(0)$

$= -3$

$N^{-1} = \frac{1}{\det A}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

$= -\frac{1}{3}\begin{pmatrix} 3 & -1 \\ 0 & -1 \end{pmatrix}$

Now that we have looked at these examples, attempt the following activity.

Activity (6.6) Inverse of a Matrix

Questions

Find the inverse of the following matrices

a) $\begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix}$ b) $\begin{pmatrix} 1 & 9 \\ -1 & 0 \end{pmatrix}$ c) $\begin{pmatrix} 1 & -4 \\ 3 & 2 \end{pmatrix}$

d) $\begin{pmatrix} -1 & 4 \\ 1 & 2 \end{pmatrix}$ e) $\begin{pmatrix} 1 & -6 \\ -4 & 2 \end{pmatrix}$

Answers

1 a) $\frac{1}{2}\begin{pmatrix} 1 & -3 \\ 0 & 2 \end{pmatrix}$ b) $\frac{1}{9}\begin{pmatrix} 0 & -9 \\ 1 & 1 \end{pmatrix}$ c) $\frac{1}{14}\begin{pmatrix} 2 & 4 \\ -3 & 1 \end{pmatrix}$

d) $-\frac{1}{6}\begin{pmatrix} 2 & -4 \\ -1 & -1 \end{pmatrix}$ e) $-\frac{1}{22}\begin{pmatrix} 2 & 6 \\ 4 & 1 \end{pmatrix}$

6.8 IDENTITY MATRIX

An identity matrix is denoted by **I**

So an identity matrix, **I=**$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Note that: If you pre-multiply or post-multiply any 2×2 matrix by an identity matrix it remains the same.

Note that: If you multiply a matrix by its inverse you get an identity matrix that is

$\mathbf{MM^{-1} = I}$

Worked Example [13]

Questions

If $N\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}=\begin{pmatrix}7 & 0\\2 & -2\end{pmatrix}$, find matrix N

Solution

Let $N=\begin{pmatrix}a & b\\c & d\end{pmatrix}$

$\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}\begin{pmatrix}a & b\\c & d\end{pmatrix}=\begin{pmatrix}7 & 0\\2 & -2\end{pmatrix}$,

$\begin{pmatrix}a+0 & b+0\\0+c & 0+d\end{pmatrix}=\begin{pmatrix}7 & 0\\2 & -2\end{pmatrix}$,

$\begin{pmatrix}a & b\\c & d\end{pmatrix}=\begin{pmatrix}7 & 0\\2 & -2\end{pmatrix}$,

$\therefore N=\begin{pmatrix}7 & 0\\2 & -2\end{pmatrix}$

Worked Example [14]

Questions

Given that $M\begin{pmatrix}-4 & -3\\5 & 3\end{pmatrix}=\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$, find matrix M

Solution

Let $M=\begin{pmatrix}a & b\\c & d\end{pmatrix}$

$\begin{pmatrix}a & b\\c & d\end{pmatrix}\begin{pmatrix}-4 & -3\\5 & 3\end{pmatrix}=\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$, multiplying matrices on the L.H.S

Using the fact that if you multiply a matrix by its inverse you get an identity matrix ($\mathbf{MM^{-1}=I}$), so we find the inverse of $\begin{pmatrix}-4 & -3\\5 & 3\end{pmatrix}$,

$\therefore M=\frac{1}{3}\begin{pmatrix}3 & 3\\-5 & -4\end{pmatrix}$.

Worked Example [15]

Question

Z is a 2×2 matrix such that $Z\begin{pmatrix}6 & 0\\0 & 6\end{pmatrix}-Z=\begin{pmatrix}-5 & 0\\5 & 10\end{pmatrix}$

Solution

First make matrix $\begin{pmatrix} 6 & 0 \\ 0 & 6 \end{pmatrix}$ an identity matrix by factoring out 6

$Z \times 6\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - Z = \begin{pmatrix} -5 & 0 \\ 5 & 10 \end{pmatrix}$, rearrange the matrices

$6Z\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - Z = \begin{pmatrix} -5 & 0 \\ 5 & 10 \end{pmatrix}$, since 6Z is multiplying an identity matrix it remains the same,

$6Z - Z = \begin{pmatrix} -5 & 0 \\ 5 & 10 \end{pmatrix}$

$5Z = \begin{pmatrix} -5 & 0 \\ 5 & 10 \end{pmatrix}$, divide both sides by 5 to get

$Z = \begin{pmatrix} -1 & 0 \\ 1 & 2 \end{pmatrix}$

Worked Example [16]

Questions

Find x and y such that $\begin{pmatrix} 3 & 7 \\ x & y \end{pmatrix}\begin{pmatrix} y & -7 \\ -2 & 3 \end{pmatrix} = \mathbf{I}$, where **I** is an identity matrix.

Solution

Multiply the 2 matrices as usual and equate them to an identity matrix

$\begin{pmatrix} 3 & 7 \\ x & y \end{pmatrix}\begin{pmatrix} y & -7 \\ -2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$\begin{pmatrix} 3 \times y \times +7 \times -2 & 3 \times -7 + 7 \times 3 \\ x \times y + y \times -2 & x \times -7 + y \times 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$\begin{pmatrix} 3y - 14 & 0 \\ xy - 2y & -7x + 3y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$,

equating corresponding values we get;

$3y - 14 = 1$,

$3y \quad = 15$,

$\therefore y = 5$ and

$-7x + 3y = 1$, but $y = 5$

$-7x + 3(5) = 1$

$-7x = -14$,

$\therefore x = 2$

Activity (6.7) Identity Matrix

Questions

1) If M is a 2×2 matrix such that M$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$=$\begin{pmatrix} 5 & 3 \\ 1 & -7 \end{pmatrix}$, find matrix M.
2) Given that H = $\begin{pmatrix} 4 & 2 \\ 0 & 3 \end{pmatrix}$ and J=$\begin{pmatrix} 0{,}25 & m \\ 0 & \frac{1}{3} \end{pmatrix}$, find the value of m which makes HJ an identity matrix.
3) Find a 2×2 matrix Y such that $\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$Y− $\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$=$\begin{pmatrix} 3 & 4 \\ -3 & 1 \end{pmatrix}$
4) Given that R is a 2×2 matrix such that R$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ +R=$\begin{pmatrix} 3 & 6 \\ 9 & 12 \end{pmatrix}$, find R.

Answers

1) M=$\begin{pmatrix} 5 & 3 \\ 1 & -7 \end{pmatrix}$ 2) $m = -\frac{1}{6}$ 3)Y=$\begin{pmatrix} 8 & 8 \\ -2 & 4 \end{pmatrix}$ 4) R= $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$

6.9 SINGULAR AND NON-SINGULAR MATRIX

Consider the following matrix$\begin{pmatrix} 3 & -2 \\ -6 & 4 \end{pmatrix}$. What is its determinant?

…… By calculation you will get the determinant of this matrix being zero.

If you proceed to find the inverse of this matrix you will get $\frac{1}{0}\begin{pmatrix} 4 & 2 \\ 6 & 3 \end{pmatrix}$ but $\frac{1}{0}$ does not exist so the inverse does not exist. Remember that any matrix which has no inverse and having a determinant of 0 is known as a singular matrix.

Worked Example [17]

Question

Given that $\begin{pmatrix} w & 6 \\ -6 & 12 \end{pmatrix}$ has a determinant of zero, find w.

Solution

First find the determinant of the matrix and equate it to zero.

Det $= ad - bc$

$0 \quad = 12w - (-6 \times 6)$

$0 \quad = 12w + 36$

$-36 \quad = 12w$ divide both sides by 12

$\therefore w = -3$

Worked Example [18]

Questions

Given that A = $\begin{pmatrix} 2x+1 & 3 \\ 1 & x \end{pmatrix}$

a) Find the determinant of A in terms of x

b) Find the values of x given that A is singular.

Solutions

a) Det A $= ad - bc$

$= x(2x+1) - 3(1)$

$= 2x^2 + x - 3$

b) $2x^2 + x - 3 = 0$, since the determinant of a singular matrix is 0

$2x^2 - 2x + 3x - 3 = 0$

$2x(x-1) + 3(x-1) = 0,$

$(x-1)(2x+3) = 0.$

$\therefore x = 1 \text{ or } x = -1\frac{1}{2}.$

Worked Example [19]

Question

If $\begin{pmatrix} -2 & p \\ p+3 & -4p \end{pmatrix}$ has no inverse, find 2 possible values of p.

Solution

$$\text{Det} = ad - bc$$

$$0 = -2(-4p) - (p(p+3)), \text{ then}$$

$$0 = 8p - p^2 - 3p,$$

$$p^2 - 5p = 0,$$

$$p(p-5) = 0,$$

$$\therefore p = 0 \text{ or } p = 5.$$

Now that we have looked at a few examples, attempt the questions in the following activity.

Activity (*6.8*) Singular Matrix

Questions

1) State why matrix$\begin{pmatrix} 12 & -6 \\ -4 & 2 \end{pmatrix}$ has no inverse?
2) Find 2 possible values of m, that makes $\begin{pmatrix} m+3 & 4m-3 \\ m & 2m+2 \end{pmatrix}$ a singular matrix.
3) Given that A=$\begin{pmatrix} 7 & 5x-2 \\ 2 & 2x+4 \end{pmatrix}$, find the value of x such that matrix A has no inverse.
4) Find 2 possible values of x such that$\begin{pmatrix} x & 3x-1 \\ 2 & 3x-1 \end{pmatrix}$has a determinant of zero.

Answers

1) it has no inverse because it has a determinant of zero.

2) $m = 6 \text{ or } m = -\frac{1}{2}$ 3) $x = -8$ 4) $x = \frac{1}{3} \text{ or } x = 2$

6.10 SIMULTANEOUS LINEAR EQUATIONS IN 2 VARIABLES

Simultaneous equations can be solved using the matrix method other than elimination, graphical or substitution methods. Let us consider the following examples.

Worked Example [20]

Question

Find the value of x and y

$\begin{pmatrix} 4 & 1 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 9 \\ 5 \end{pmatrix}$

Solution

In order for us to proceed with the calculation we need to find the inverse of $\begin{pmatrix} 4 & 1 \\ 3 & -1 \end{pmatrix}$, the inverse of the matrix is$-\frac{1}{7}\begin{pmatrix} -1 & -1 \\ -3 & 4 \end{pmatrix}$, then we pre-multiply the matrix equation with the inverse

$-\frac{1}{7}\begin{pmatrix} -1 & -1 \\ -3 & 4 \end{pmatrix}\begin{pmatrix} 4 & 1 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}= -\frac{1}{7}\begin{pmatrix} -1 & -1 \\ -3 & 4 \end{pmatrix}\begin{pmatrix} 9 \\ 5 \end{pmatrix}$, remember that if you multiply a matrix by is inverse you get an identity matrix so,

$\begin{pmatrix} x \\ y \end{pmatrix}= -\frac{1}{7}\begin{pmatrix} -1 & -1 \\ -3 & 4 \end{pmatrix}\begin{pmatrix} 9 \\ 5 \end{pmatrix}$, multiplying matrices on the R.H.S we get;

$\begin{pmatrix} x \\ y \end{pmatrix}= -\frac{1}{7}\begin{pmatrix} -14 \\ -7 \end{pmatrix}$

$\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$\therefore x = 2 \text{ and } y = 1.$

Worked Example [21]

Question

Solve the following equations using the matrix method

$2x + 3y = 6 \text{ and } 6x + 4y = 5.$

Solution

First arrange the equations in matrices

$\begin{pmatrix} 2 & 3 \\ 6 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 6 \\ 5 \end{pmatrix}$, solve the equation by finding the inverse of the matrix on the L.H.S

$\frac{-1}{10}\begin{pmatrix} 4 & -3 \\ -6 & 2 \end{pmatrix}\begin{pmatrix} 2 & 3 \\ 6 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \frac{-1}{10}\begin{pmatrix} 4 & -3 \\ -6 & 2 \end{pmatrix}\begin{pmatrix} 6 \\ 5 \end{pmatrix}$,

$\begin{pmatrix} x \\ y \end{pmatrix}=\frac{-1}{10}\begin{pmatrix} 9 \\ -26 \end{pmatrix}$

$\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} -0{,}9 \\ 2{,}6 \end{pmatrix}$

$$\therefore x = -0{,}9 \text{ and } y = 2{,}6$$

Activity (*6.9*) Application of Matrices

Questions

By referring to the examples above try to attempt the following questions

1) $2x + 3y = 11$
   $3x - 5y = -12.$

2) $8x + 15y = 11$
   $4x - y = -3$

Answers

1) $x = 1$ and $y = 3$ 2) $x = \frac{-1}{2}$ and $y = 1$

REFLECTION

- You are only able to add or subtract matrices of the same order
- When you are multiplying matrices, use a row by column technique where only rows of the first matrix multiply columns of the second matrix
- There are special matrices namely singular, identity and null matrix.

6.11 Summary

The unit showed that matrices are used as a way of storing information. We have seen that the order of a matrix is given by counting number of rows followed by number of columns of a given matrix. The unit also looked at the determinant of a 2x2 matrix=ad − bc, inverse of a 2 × 2 matrix = $\frac{1}{\det}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$, Identity matrix which is denoted by I and represented as I = $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and that singular matrix has no inverse. We have also looked at the application of matrices in solving linear simultaneous equations.

6.12 Further Reading

1. Macrae, M, Madungwe, L and Mutangadura, A (2017) New General Mathematics Book 3, Pearson Capetown.
2. Macral, M, Madungwe, L and Mutangadura, A (2017) New General Mathematics Book 4, Pearson Capetown.
3. Meyers, C; Graham, B; Dawe, L (2004) Mathscape Working Mathematically, 9th Ed, MacMillan
4. Pimentel R and Wall T (2011) International Mathematics, Hodder UK
5. Rayner D (2005) Extended Mathematics. Oxford New York

6.13 Assessment Test

1) It is given that P=$\begin{pmatrix} 3 & 5 \\ 4 & x \end{pmatrix}$ and Q=$\begin{pmatrix} -2 \\ 3 \end{pmatrix}$

   (i) Find PQ in terms of x

   (ii) Find the value of x that makes $|P| = 7$

   (iii) Hence write down P^{-1}

2) Given that A=$\begin{pmatrix} 3 & 2 \\ 4 & 3 \end{pmatrix}$ B=$\begin{pmatrix} 1 & 3 \\ 2 & 0 \end{pmatrix}$find

   a) A−2B

   b) AB

   c) A 2× 2 matrix X such that AX=**I**, where **I** is an identity matrix.

3) $R=\begin{pmatrix} -1 & x \\ 3 & 2x \end{pmatrix}$ $S=\begin{pmatrix} 5 & -4 \\ -2 & 3 \end{pmatrix}$, find

a) The value of x if the determinant of R is 20.

b) The inverse of S

UNIT 7 - GEOMETRY 1

CONTENTS

7.1 INTRODUCTION

What you are going to learn in this unit is a continuation of the work you covered in Level one. You are going to learn about different types of angles in different types of shapes. the unit is also going to cover practical aspects on direct using angles.

OBJECTIVES

After going through this unit you should be able to:

- identify types of angles such as right, straight, acute, obtuse and reflex angles,
- vertically opposite angles,
- state and use properties of n-sided polygons,
- identify special names given to n-sided polygons,
- identify, interpret and apply concepts points, line, segments, parallels and perpendiculars,
- identify lines of symmetry and order of symmetry of plane figures or shapes,
- use the exterior and interior angle properties of polygons.

KEY TERMS

Polygon: - Is any plane figure with straight edges or sides.

Regular: -length and angles are of the same size.

Vertex: - Point where two lines of a triangle meet, vertices when many.

Transversal line: - Line which crosses or cut parallel lines.

Line of symmetry: - A line which divides into two equal parts it is also called a mirror line.

Convex: - Closed.

TIME: You should not spent more than 10 hours on this unit.

STUDY SKILLS

The key to understanding geometry is to familiarise yourself with the properties of angles which should be followed by practice. Once this is done, then the learning of all Geometry and other related concepts will become easy.

7.2 POINTS, LINES AND ANGLES

There are many angle relationships in geometry. The most common relationships are explained as follows:

7.2.1 Vertically opposite angles are equal

Fig 7.1 below shows that when two straight lines intersect we get vertically opposite angles. **a** is vertically opposite to **c** and **b** is vertically opposite to **d**.

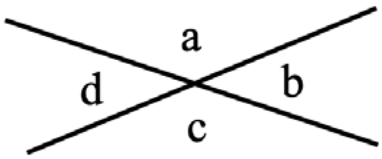


Fig 7.1 – vertically opposite angles

7.2.2 Angles on a straight line add up to 180°

Fig 7.2 shows two angles which are adjacent. They are on a straight line and they add up to 180°. Thus x + y = 180°.

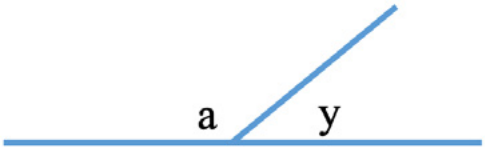


Fig 7.2 – angles on a straight line

7.2.3 The sum of all angles around a point add up to 360°

In fig 7.3 below we have angles at a point **a + b + c = 360°.**

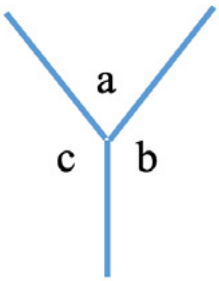


Fig 7.3 – angles around a point

7.2.4 Corresponding angles are equal

Fig 7.4 below shows two angles **a** and **b**. The two angles are equal and are known as **corresponding angles**.

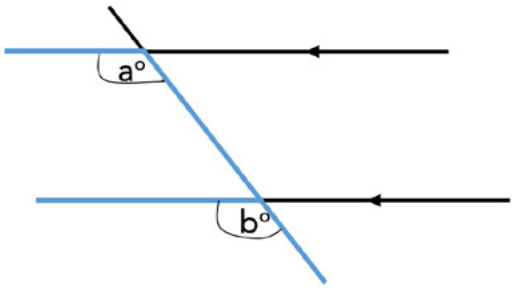


Fig 7.4 –Corresponding angles

Corresponding angles can be found by looking for an 'F' formation in the diagram. Fig 7.5 is a variation of the 'F' formation diagram.

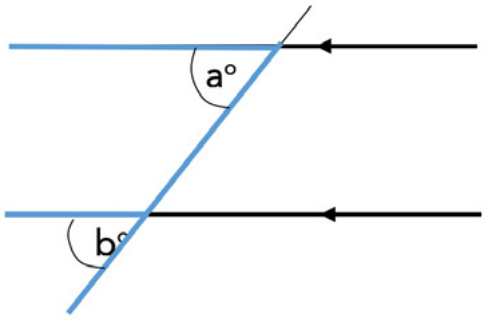


Fig 7.5 – Corresponding angles

7.2.5 Alternate angles are equal

Fig 7.6 shows two angles **a** and **c**. these angles are formed by a line which intersects two parallel lines. Angle **a** and **c** are equal. They are known as alternate angles.

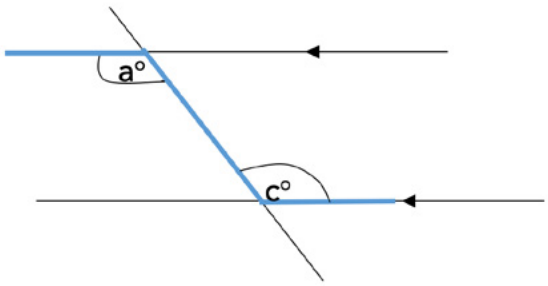


Fig 7.6 – Alternate angles

Alternate angles can be found by looking for a 'Z' formation in the diagram. Fig 7.7 is a variation of the 'Z' formation depicting alternate angles.

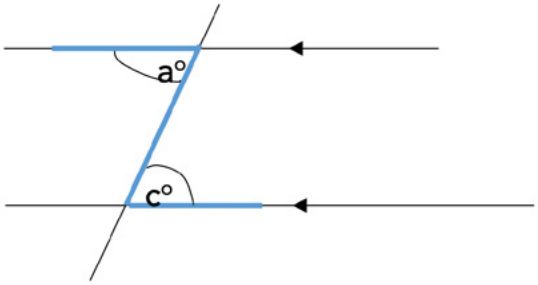


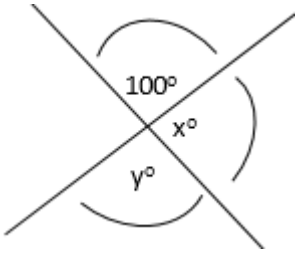
Fig 7.7 – Alternate angles

Now that we have gone through some basic properties of angles, let us look at some worked examples.

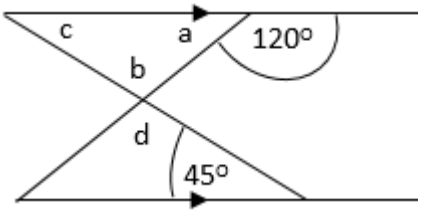
Worked Example [2]

Questions

a. Find the values of **x** and **y** in the diagram below:



b. Use the diagram below to find lettered angles.



Solutions

a. 100° + x = 180°

x = 180° – 100°

x = 80°

y = 100 (vertical opposite angles)

b. **a** + 120° = 180° (angles on a straight line)

a = 180° – 120°

a = 60°

c = 45° (**c** is alternate to 45°)

a + **c** + **b** = 180° (sum of angles in triangle)

= 60° + 45° + **b** = 180°

180° – 105°

b = 75°

d = 75° (vertical opposite angle are equal)

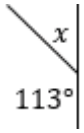
Now that we have gone through a few examples, I think you are now ready for an exercise. Attempt the following activity.

Activity (*7.1*) Types of angles

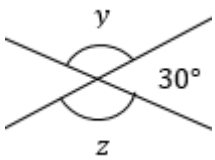
Questions

Using angle properties, calculate the value of the unknown angles in each of these questions.

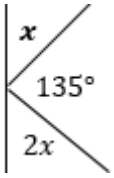
1 (a)



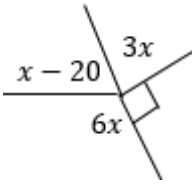
(b)



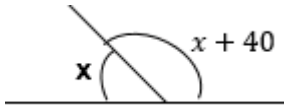
(c)



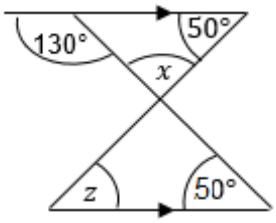
2 (a)



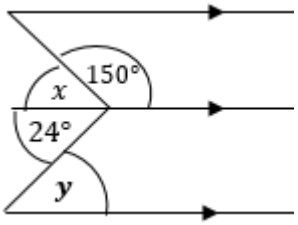
(b)



3 (a)



(b)



Answers

1. a) x = 67° b) y = 150°, z = 150° c) x = 15°
2. a) x = 29° b) x = 70°
3. a) x = 80°, z = 50° b) x = 30°, y = 24°

7.3 LINES OF SYMMETRY

The line of symmetry can be defined as the axis or imaginary line that passes through the centre of the shape or object and divides it into identical halves.

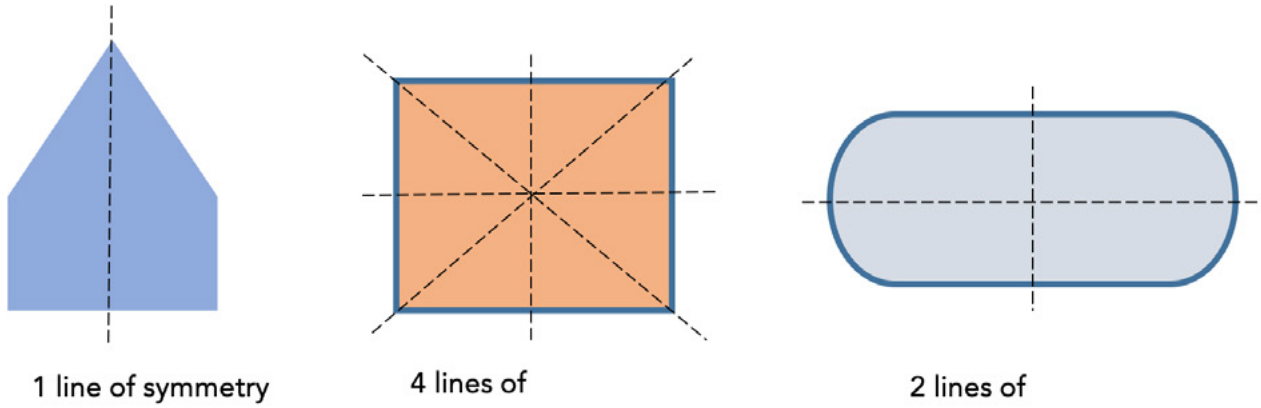


Fig 7.8 Lines of Symmetry

7.3.1 Rotational or Point Symmetry

A two-dimensional shape has rotational symmetry if, when rotated about a central point, it fits its outline. Rotation can be clockwise or anticlockwise. The number of times it fits its outline during a complete revolution is called the order of rotational symmetry.

An equilateral will fit exactly onto itself3 times before it comes to its original position. Therefore its order of rotational symmetry is 3.

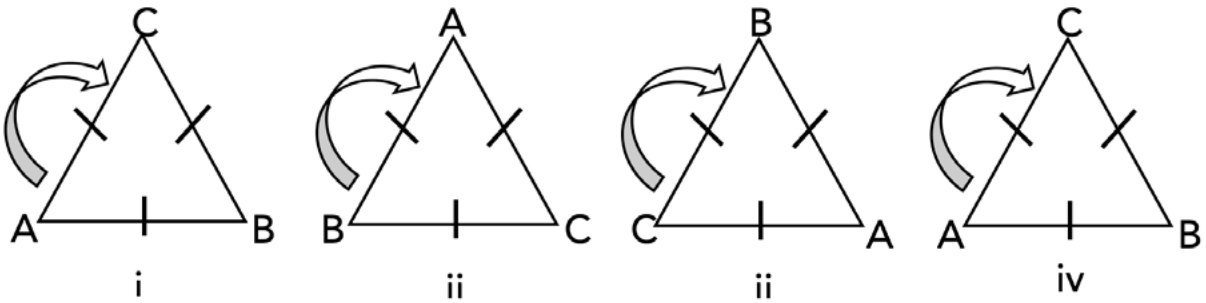


Fig 7. 8 – rotational symmetry of a triangle

The rotation is clockwise and it is rotating through 60°, check the positions of A until it gets back to its starting point.

A rectangle will behave different from an equilateral triangle. In the first rotation, the rectangle will be vertical and in second time it will be horizontal and fit its self. Hence it has order 2 of rotation in order for it to fit in itself.

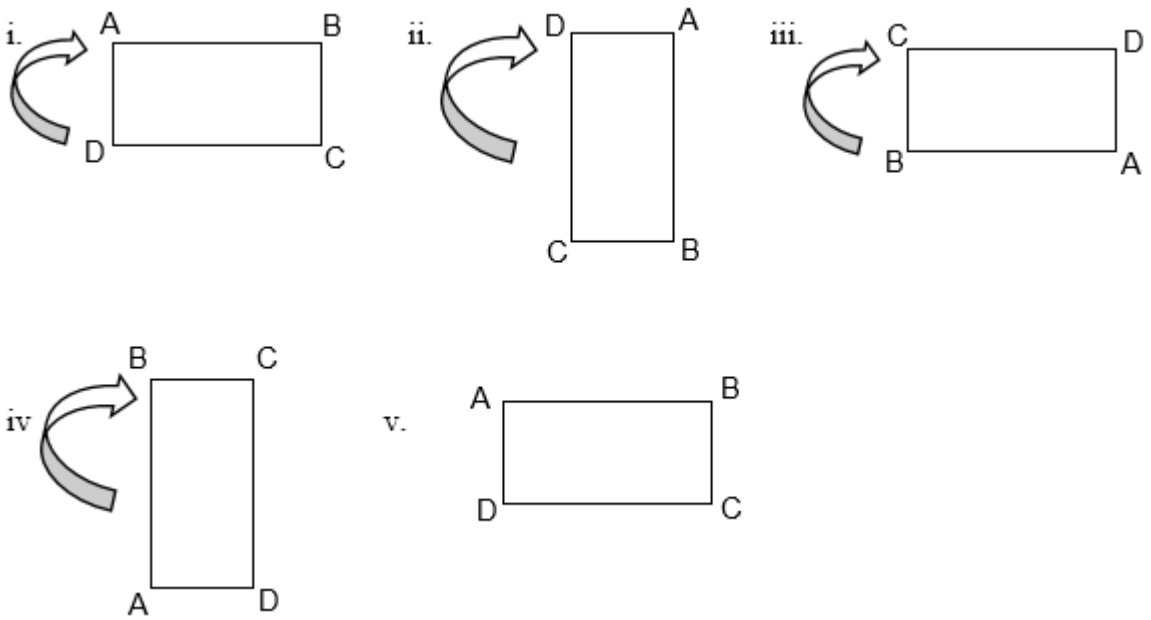


Fig 7. 9 – Rotational symmetry of a rectangle

Activity (7.2) Symmetry

Questions

Complete table 7.1 below. The first two have been done for you.

TIP: Make or draw these shapes, fold the shape to see the lines of symmetry and rotate shape to see order of rotation.

Table 7.1 – Lines of symmetry

SHAPE	LINE OF SYMMETRY	ORDER OF ROTATIONAL SYMMETRY
Rectangle	2	2
Kite	1	2
Rhombus		

Circle		
Rectangle		
Parallelogram		
Regular hexagon		
Isosceles triangle		
Equilateral triangle		

Answers

SHAPE	LINE OF SYMMETRY	ORDER OF ROTATIONAL SYMMETRY
Rectangle	2	2
Kite	1	2
Rhombus	2	2
Circle	Infinity	Infinity
Rectangle	2	2
Parallelogram	0	2
Regular hexagon	6	6
Isosceles triangle	1	-
Equilateral triangle	3	3

7.4 POLYGONS

The meaning of the word polygon is defined when the word is separated as follows:

Poly: - means many.

Gon: - means angles.

Therefore when the two are joined together thus polygon means many angles.

Therefore a triangle is a three sided polygon and a quadrilateral is a four sided polygon. **Tri** means three (3) and **Quad** means four (4). Therefore polygons are named according to number of angle or sides. Hence we have polygons as shown in table 7.2:

Table 7.2 – Polygons

Triangle	**3 sided / angles**	**1 triangle**
Quadrilateral	4 sided/ angles	2 triangles
Pentagon	5 sided/ angles	3 triangles
Hexagon	6 sided/ angles	4 triangles
Heptagon	7 sided/ angles	5 triangles
Octagon	8 sided/ angles	6 triangles
Nonagon	9 sided/ angles	7 triangles
Decagon	10 sided/ angles	8 triangles

7.4.1 Properties of Some Common Polygons

Square: - All sides are equal, opposite sides are parallel and all angles are right angles. Diagonals are equal and bisect each other at 90° at the centre.

Rhombus: - All sides are equal, opposite sides are parallel, opposite angles are equal, one pair of its sides is acute and the other pair is obtuse.

Rectangle: - All angles are right angles, diagonals are equal, opposite sides are equal and parallel and diagonals bisect each other at the centre.

Parallelogram: - Opposite sides are equal and parallel, diagonals bisect each other at 90°, one pair of its angles is acute, diagonals bisect the angles and another pair of its angles is obtuse.

Trapezium: - One pair of its sides is parallel and at least one angle is obtuse.

Kite: - Its adjacent sides are equal, one pair of its opposite angles are equal and longer diagonal bisect shorter at 90°.

The properties of above shapes will help you to calculate angles and sides. Know them and use them.

7.4.2 Regular Polygons

A regular polygon is distinctive in that all its sides are of equal length and all its angles are of equal size.

Fig 7.10- Regular polygons

Regular polygons have two kinds of angles as shown in fig 7.11 below:

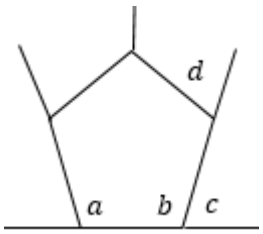


Fig 7. 11 – Angles on a regular polygon

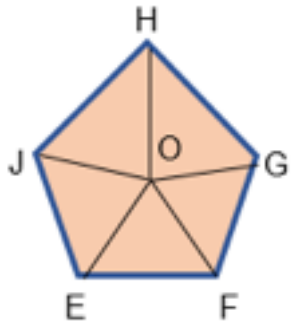
a and **b** are examples of interior angles **c** and **d** are examples of exterior angles. **b**and **c** are adjacent to each other and they add up to 180°. The exterior angles of any polygon always add up to 360°.

After going through the examples given, you should now be prepared to answer questions on the following activity.

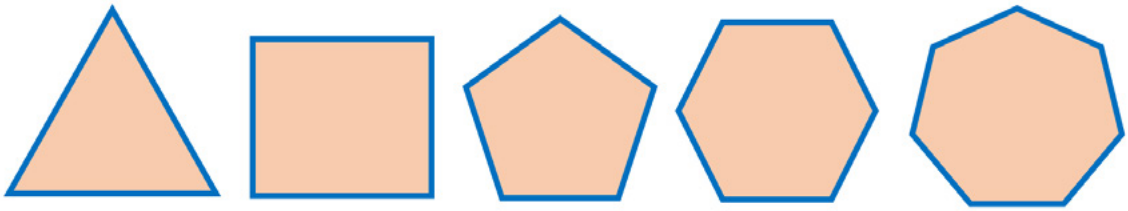
Activity (7.3) Properties of Polygons

Questions

1. The diagram below shows a regular pentagon EFGHJ with centre O.
    a. How many angles are at the centre O?
    b. Calculate the side of each angle at the centre.
    c. Find the sum of its angles at O.
    d. Calculate angles OEF and OFE.
    e. What name is given to triangle EOF?



2. How many lines of symmetry in each of the following polygons?



(a) (b) (c) (d) (e)

Answers

1. a) 5 b) 72° c) 360° d) 54° e) Isosceles triangle
2. a) 1 b) 4 c) 1 d) 6 e) 8

7.4.3 Sum of Interior and Exterior Angles of Polygons

Interior Angles of a Polygon

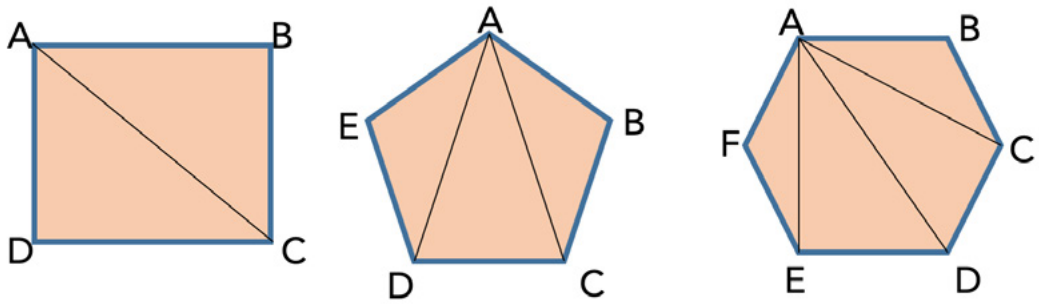


Fig 7.12 – Interior angles of a polygon

If a straight line is drawn from each vertex to vertex A, triangles will be produced. The number of triangles produced is always 2 less than the number of sides the polygon has. For example, if there are n sides, there will be (n – 2) triangles produced.

Since there are 2 triangles less than the number of sides of the polygon, the sum of interior angles any n-sided polygon is (n – 2) × 180°. You can check this by working out the sum of the angles of a triangle and rectangle which you already know.

Worked Example [2]

Questions

Find the sum of the interior angles of

1) a triangle
2) a rectangle

Solutions

1) Thus the sum of angles in a triangle = (n – 2) ×180°

$= (3 - 2) \times 180°$

$= 1 \times 180°$

$= 180°$ **true, is it not?**

2) Also the sum of angles in a rectangle = $(n - 2) \times 180°$

$= (4 - 2) \times 180°$

$= 2 \times 180°$

$= 360°$

∴This formulae is true for any n-sided polygon.

Activity (7.4) Sum of Interior Angles

Questions

Calculate the sum of the interior angles of a regular :

a) Pentagon. b) Heptagon. c) Octagon.

Answers

a) 540° b) 900° c) 1080°

Let us look into some more examples, this time we would like to look at how to find the number of sides of a polygon given the sum of its interior angles.

Worked Example [3]

Question

The sum of interior angles of a regular polygon is = 1440°. How many sides does the polygon has?

Solution

Using the formulae

Sum of interior angles = $(n - 2) \times 180°$

Thus,

$1440° = (n - 2) \times 180°$

$1440° = 180n - 360°$

$1440° - 360° = 180n$

$1800° = 180°n$

$n = \frac{1800°}{180°}$

$n = 10$

Therefore the polygon has 10 sides.

Remember:

Sometimes you may see the formula $(2n – 4) \times 90°$. Do not be confused by that because

$2n – 4$ is the number of right angles in the polygon. You can also realise that if you divide

$(2n – 4)$ by 2 and multiply 90° by 2 then you will go back to $(n – 2) \times 180°$.

Activity (7.5) Sum of Interior Angles

Questions

1) Calculate the sum of interior angles by completing Table 7.3, the first one been done for you:

Table 7.3 – Sum of interior angles

Polygon	Number of sides	Number of triangles	Sum of angles at O	Sum of angles
Pentagon	5	5	360°	$5 \times 180 – 360 = 540°$
Hexagon				
Heptagon				
Octagon				

2) The sum of the angles of a polygon is 1980°. How many sides does the polygon has?
3) A polygon has 16 sides. Calculate the sum of its interior angles.

Answers

1)

Polygon	Number of sides	Number of triangles	Sum of angles at O	Sum of angles
Pentagon	5	5	360°	5 ×180–360=540°
Hexagon	6	6	360°	6 ×180–360=720°
Heptagon	7	7	360°	7 ×180–360=900°
Octagon	8	8	360°	8 ×180–360=1080°

2) 9

3) 2520°

Exterior Angles of a Polygon

The sum of the regular exterior angles of any polygon is 360°.

Worked Example [4]

Questions

1) Calculate the exterior and interior angles of regular nonagon.
2) How many sides has a regular polygon if each interior angle is 150°.

Solutions

1) Nonagon has 9 sides. Hence:

Exterior angle = $\frac{360°}{9}$

= 40°

Interior angle = 180° - 40°

= 140°

2) To find the number of sides

Interior angle = 150°

Exterior angle = 180° - 150°

= 30°

Therefore number of sides = $\frac{360°}{30°}$

= 12

Hence the polygon has 12 sides.

Remember: The interior angle of any polygon is adjacent to the exterior angle and when you add the two you get a straight angle which is 180°. Since regular means equal we divide 360° by number of sides.

Now that we have looked at some examples, attempt the questions on the following activity.

Activity (7.6) Sides and angles of regular polygons

Questions

1) Find the number of sides of a regular polygon, if the exterior angle is 36°.
2) Calculate the number of sides of a regular polygon whose interior angles are:
   a. 168°. b. 156°. c. 144°
3) Find the number of sides of a polygon if its interior angles add up to 1620°.
4) A hexagon has the following angles: x, 2x, 3x, 4x, 5x and 6x calculate:
   a. The value of x. b. Size of exterior angle.
5) Calculate the number of sides of a regular polygon if the exterior angle is:
   a. 24°. b. 90°. c. 15°.
6) Calculate the number of sides of a regular polygon whose interior angle is:
   a. 140°. b. 150°. c. 120°.

Answers

1) 10
2) a) 30 b) 15 c) 10
3) 7 sides
4) a) 34.3° b) 60°
5) a) 15 b) 4 c) 24
6) a) 9 b) 12 c) 6

7.5 Summary

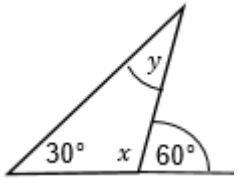
Polygons are named according to number of their sides. A regular polygon has all its sides with the same length and its angles are of the same size. A formulae for calculating the sum of the interior angles of an n-sided polygon was seen to be (n – 2) x 180° or (2n – 4) x 90° right angles. The same formulae can be used to find number of sides of an n-sided polygon. The sum of the exterior angles of any polygon add up to 360°.

7.6 Further Reading

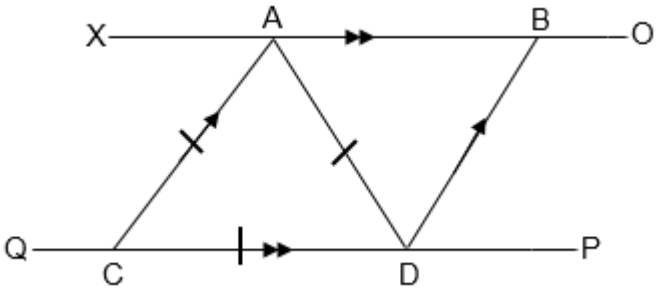
1. Macrae, M, Madungwe, L and Mutangadura, A (2017) New General Mathematics Book 3, Pearson Capetown.
2. Macral, M, Madungwe, L and Mutangadura, A (2017) New General Mathematics Book 4, Pearson Capetown.
3. Meyers, C; Graham, B; Dawe, L (2004) Mathscape Working Mathematically, 9th Ed, MacMillan
4. Pimentel R and Wall T (2011) International Mathematics, Hodder UK
5. Rayner D (2005) Extended Mathematics. Oxford New York

7.7 Assessment Test

1) Calculate the sizes of angle marked with a letter on the diagram below.



2) What do you know about:
   a) Supplementary angles.
   b) Allied angles.
   c) Complementary angles.
3) A 7 sided regular polygon:
   a) What is it called?
   b) Find the sum of its:
      i. Interior angles.
      ii. Exterior angles.
4) Calculate the number of sides of a regular polygon whose:
   a) Exterior angle is 36°.
   b) Interior angle is 140°.
5) Find the size of each interior angle of a 20-sided regular polygon.
6) Using Fig 7.14 calculate:
   a) Angle BDP.
   b) Angle XAQ.
   c) What name is given to triangle ABD

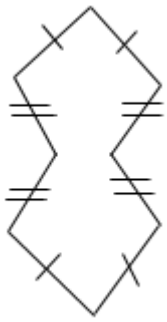


7) A hexagon has (2x)°, (12 + x)°, (4 + 3x)° and the other 3 are (x + 10)° each. Calculate the value of x .

8) Use fig 7.15 to answer:

a) What order of rotational symmetry of the shape?

b) How many number of lines of symmetry of the shape?



Answers

1) x = 120°, y = 30°
2) a) Add up to 180° b) Add up to 180°, c) add up to 90°.
3) a) Heptagon, b) i. 900°, ii. 360°
4) a) 10 b) 9
5) 162°
6) a) 60° b) 120° c) Isosceles.
7) 66,1°
8) a) 2 b) 2

UNIT 8 - GEOMETRY II

CONTENTS

8.1 INTRODUCTION

Circle Geometry is a very simple and enjoyable topic. In this topic, we are going to discover the relationship between angles inside and outside a circle. There are rules or theorems you should know and once you have understood these theorems you will be able to apply them to solve problems in circle geometry.

OBJECTIVES

After going through this unit you should be able to:

- Use facts about parts of the circle to show relationships between angles and sides.
- Apply circle theorems to calculate lengths of lines and angles by using:
  - Angle at the centre of a circle and angles subtended at the circumference,
  - Angles in the same segment
  - Angles in the semi-circle,
  - Angles in the alternate segments,
  - Angles in a cyclic quadrilateral.

KEY WORDS

Diameter: - Is a line joining two points on the circumference passing through the centre of that circle.

Segments: - Are two parts of the circle cut by a chord.

Arc: - Is a part of the circumference or an incomplete circle.

Chord: - Is a line joining two points on a circle but not passing through the centre of that circle.

Subtend: - Is an angle formed by two lines meeting at a point on the circumference from each end of a chord.

Theorem: - Is a proven law or fact or rule.

TIME

You are advised not to spend more than 10 hours in this unit.

STUDY SKILLS

This topic builds on the topic on Geometry 1 that was covered in Unit 7. One of the most important thing to note in the learning of Mathematics is that practice is the key to mastery of all mathematical concepts. You need to solve as many problems in Geometry 2 as possible for you to grasp all the concepts in this topic.

8.2 PARTS OF A CIRCLE

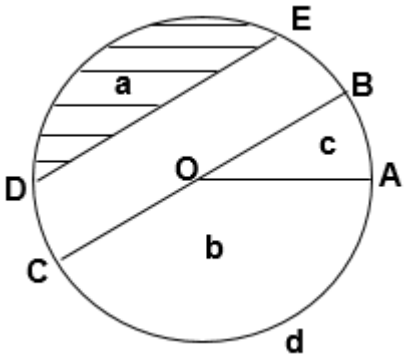
In Level 1 we learnt about the parts of a circle. Do you still remember the names given to the various parts of a circle?

Let us look at a recap by completing the following activity.

Activity (*8.1*) Parts of a circle

Questions

1. Name the parts of a circle in the diagram below.



2. Name the line:
   i. OA.
   ii. BC.
   iii. DE

Answers

1. a) segment b) sector c) Sector.
2. i. OA Radius. ii. BC Diameter. iii. DE Cord.

8.3 CIRCLE THEOREMS

8.3.1 Angles in the semicircle are right-angle.

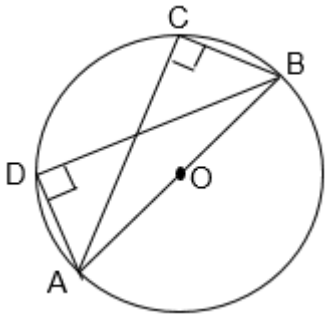


Fig 8.1 – angles in a semicircle

In Fig 8.1 O is the centre and AB is a diameter of the circle. If lines are drawn from A and B to meet at C the angle formed (subtended) is 90° (right angle). Similarly if the two lines from A and B meet at D again 90° is formed.

8.3.2 Angle on a straight line is 180°.

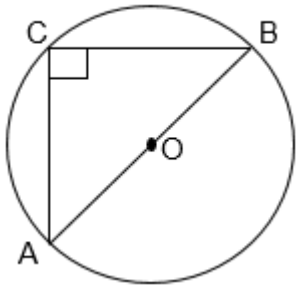


Fig 8.2 – Angle on a straight line

Angle at the centre is twice the angle at the circumference, that is, the angle subtended at the circumference is half that subtended at the centre. Have you noticed that Theorem 2 is related to Theorem 1?

8.3.3 Angles in the same segment are equal.

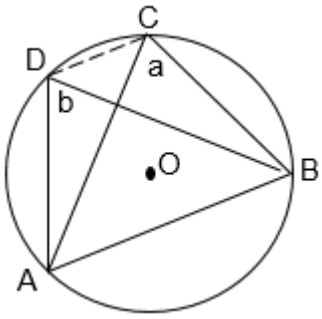


Fig 8.3 – Angles in the same segment

Angles in the same segment of a circle are equal. Angle ADB and angle ACB are in the same segment hence they are equal.

Now that we have gone through the first three circle theorems, you should now be ready to attempt the following activity.

Activity (*8.2*) Circle Theorems

Questions

1. In the following diagram, O is the centre of the circle.
   i). State angles which are equal in Fig 8.4 and give reasons in each case.
   ii). Angles BEC = 20° calculate angle **b**.
   iii). What is the size of **a.**

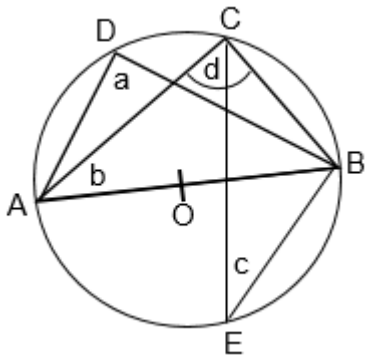


Fig 8.4

2. In Fig 8.5 below it is given that XZY = 40°,
   i. Calculate angle **a** and **b** give reasons**.**
   ii. If XO is 6cm what is the length of YO give reason.
   iii. What is the size of angle OXY and OYX.

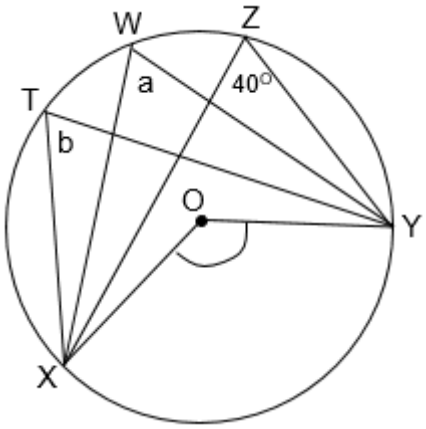


Fig 8.5

TIP: Draw line XY first before attempting iii.

Answers

1. i. a = d angles in the same segment.

   ii. b = 20°.

   iii. a = 90° angles subtended on the circumference.

2. i. a = b = 40°

   ii. YO = 6 cm (XO = YO radius of same circle)

   iii. OXY = 50°, OYX = 50°.

8.3.4 The angle subtended at the centre of the circle by an arc is twice the size of the angle on the circumference subtended by the same arc.

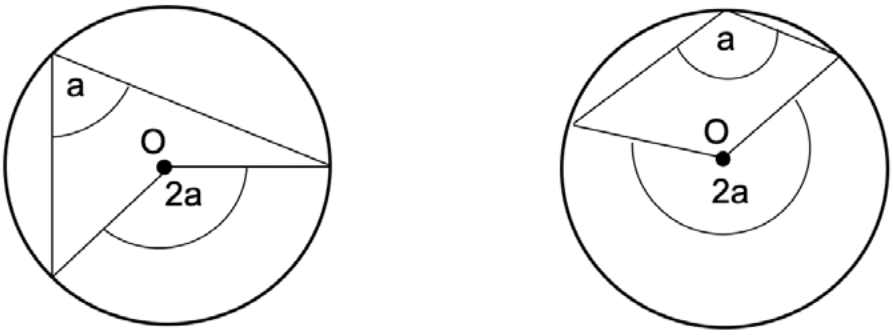


Fig 8.6 – Angle subtended at the centre

8.3.5 Opposite angles of a cyclic quad add up to 180

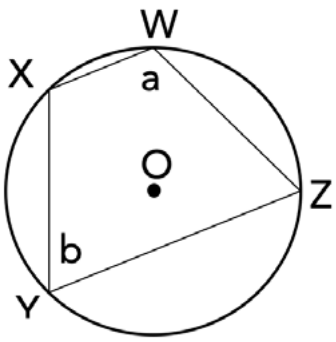


Fig 8.7 – Opposite angles of a cyclic quad

A cyclic quadrilateral is a rectangle that has all its four vertices on the circumference of a circle. Angle a + b = 180°, therefore opposite angles of a cyclic quad add up to 180° (supplementary).

8.3.6 Exterior angle of a cyclic quad is equal to the interior opposite angle.

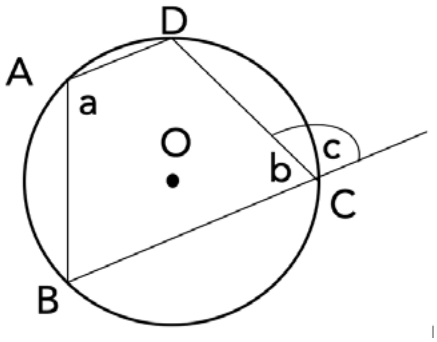


Fig 8.8 – Exterior angle of a cyclic quad

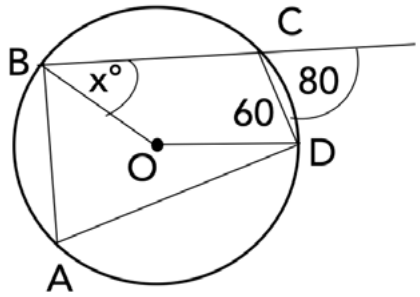
Since **a + b = 180°** (supplementary)**, b + c = 180°** (angles on a straight line), it follows that **a = c**. therefore the exterior angle of a cyclic quad is equal to the interior opposite angle.

After going through these theorems, let us see an example of how these theorems are applied.

Worked Example [1]

Question

The diagram shows a cyclic quadrilateral in a circle centre O. By applying some theorems you have learnt, calculate angle x°.



Solution

Step 1

Applying angles on a straight line.

BCD = 180° - 80°

= 100°

Step 2

BAD, exterior angle of cyclic quad is equal to the opposite interior angle.

i.e BAD = 80°

Step 3

Angle BOD = 2 x 80° (angles at the centre is twice the angles on the circumference)

= 160°

Step 4

Required angle = 360° - (100 + 160° + 60°) (sum of the angles in a cyclic quadrilateral)

= 360°- 320°

= 40°

Activity (*8.3*) Circle Theorems

Questions

1) Use the following diagram to answer the following questions:
   a. Which arc subtends angle WUX.
   b. Identify an angle equal to angle WUX.
   c. Which arc subtends angle YXZ.
   d. Identify an angle equal to angle YZW.
   e. Which arc subtends angle XWY?

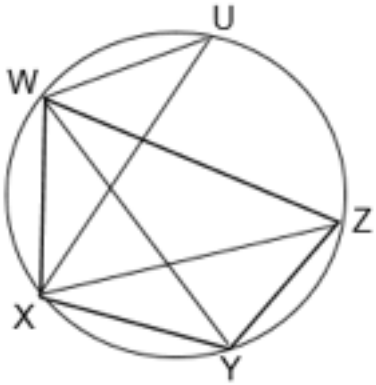
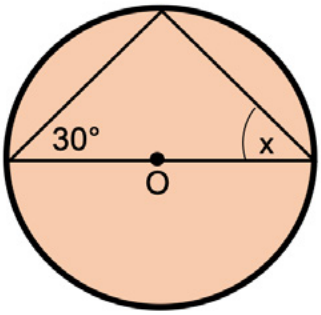


Fig 8. 1

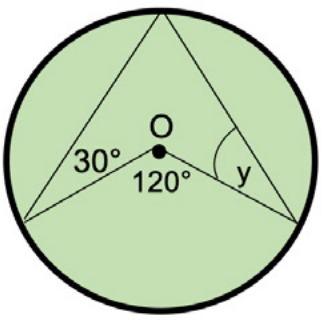
2) Calculate the marked angle in each of the following. O on the diagram indicates centre of circle.

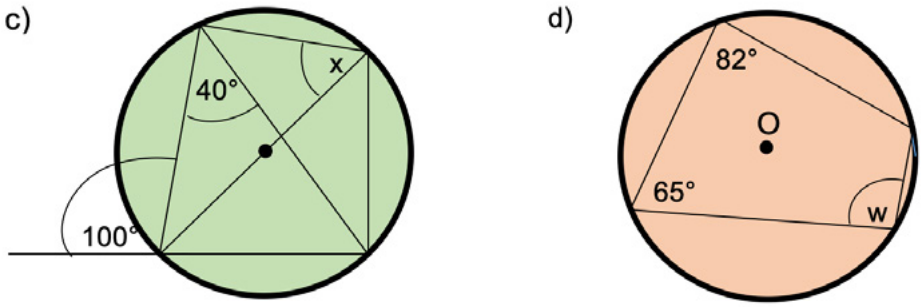
TIP: Sketch the diagram

a)



b)





Answers

1. a) WX. b) WZX and WYX. c) YZ. d) YUW e) XY.
2. a) x = 60°. b) y = 90°. c) x = 60°. d) w = 98°.

Now that you have gone through the above activity, let us continue with circle theorems.

8.3.7 The angle between a tangent at a point and the radius to the same point on the circle is a right angle.

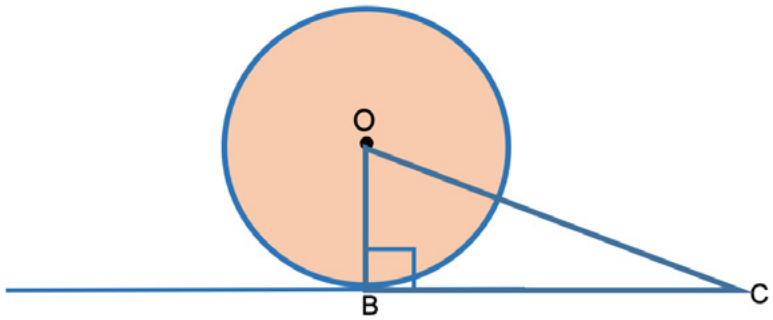


Fig 8. 2

The angle between a tangent BC and a radius OB in a circle is a right angle.

8.3.8 Angles in the alternate segments are equal.

Do you remember what a chord is? It is any straight line drawn across a circle and then it divides the circle into segments. The following diagram shows chord BC which divides the circle into two segments. Angle BCY and angle BAC are angles in alternate segments because they are in opposite segments.

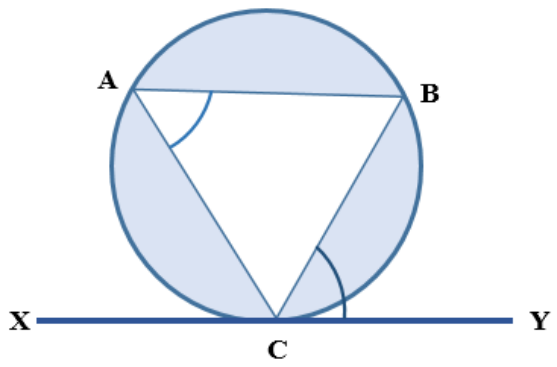


Fig 8.12

The alternate segments theorem states that an angle between a tangent and a chord through a point of contact is equal to the angle in the alternate segment. This means that angle BCY and BAC are equal.

8.3.9 Equal chords subtends equal angles.

AB and AD are two equal chords. AB subtends angle ACB, this angle is equal to angle ACD which is subtended by chord AD.

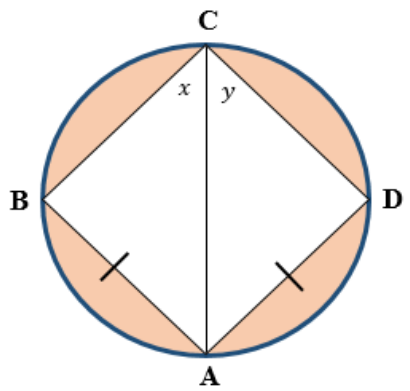


Fig 8. 3

Since the chords are equal, this means that angle x and angle y are equal.

angle x = angle y

8.3.10 Tangents from same external point to a circle are equal.

Fig 8.14 shows what is meant by theorem 9: - AT = AB. The two tangents are starting from A to the circle at T and B.

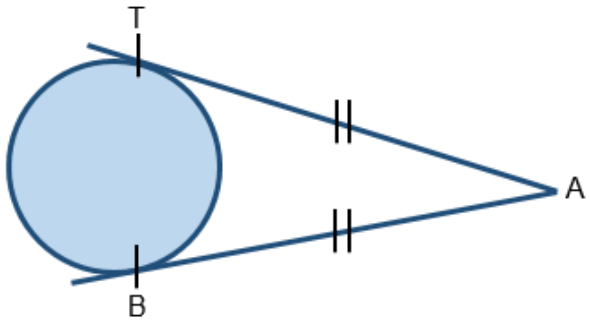


Fig 8. 4

Worked Example [2]

Question

The fig 8.15 shows circle centre O, ZX and ZY are tangents. Given that angle OXB = 20°, you are required to find angle XZY.

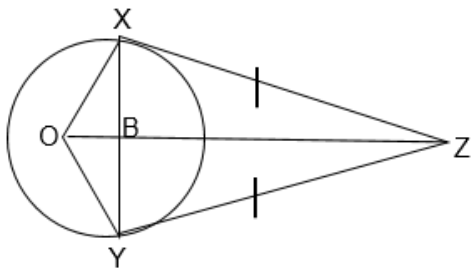


Fig 8. 5

Solution

You must remember that OX and XZ meet at 90°. Why? Yes theorem on radius and a tangent meet at 90°.

Therefore $ZXB = 90° - 20° = 70°$

ZX = ZY (tangents to a circle)

Therefore $\triangle$ ZXY = is an isosceles

ZXY = ZYX (base angles of isosceles △)

Therefore XZY = $180° - (2 \times 70°)$

$= 180° - 140°$

$= 40°$

Worked Example [3]

Question

In fig 8.16 BDC is a circle BC is a chord AB and AC are tangents angle BAC = 50°. Calculate angle BDC.

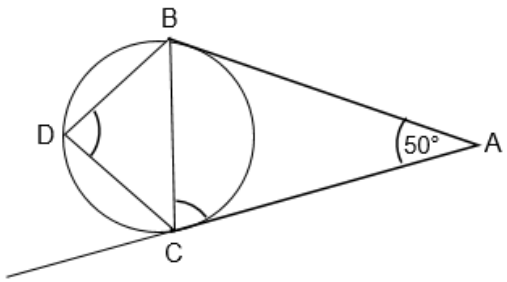


Fig 8. 6

Solution

△ ABC is isosceles. (Tangents are equal)

BCA = $\frac{1}{2}(180° - 50°)$

$= \frac{130}{2}$

$= 65°$

∴ BDC = 65°△ (∠s in the alter segment)

Worked Example [4]

Question

Fig 8.17 shows MHJ a tangent and circle LHK. Angle LHK and LKH are given, calculate KHJ.

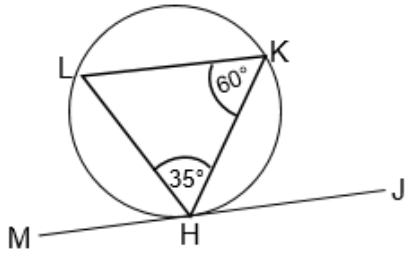


Fig 8. 7

Solution

HLK = 180° - (60° + 35°) (Sum of ∠s in Δ)

= 180° - 95°

= 85°

∴ KHJ = 85° (∠s in the alter segment)

Activity (*8.4*) Circle theorems

Questions

1) In fig 8.18 PT is a tangent, ABP = 52°, is semi-circle, O is centre. Calculate:
    a. BOP.
    b. BAP.
    c. APB.

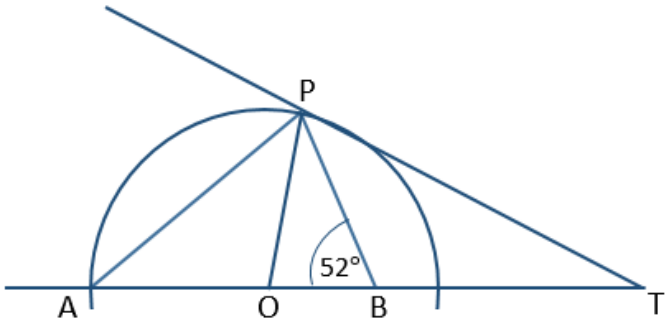


Fig 8.18

2) Fig 8.19 shows a tangent ABC to a circle BDEF. FB and DB are chords and ABF = 40°. Name five angles in fig 8.19, which are equal to 40° (give reason where possible).

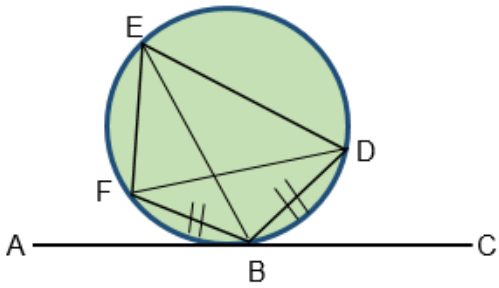


Fig 8.19

3) Use fig 8.20 to calculate required angle when given angles as shown below:
   a. If XCB = 55°, calculate XZB.
   b. XBZ = 60°, BXZ = 54°, calculate XCB.
   c. XBZ = 30°, ZBA = 85°, calculate BXZ.
   d. ZXY = 45°, ZBA = 76°, calculate XCB.
   e. XZB = 56°, ZBA = 82°, calculate ZXY.

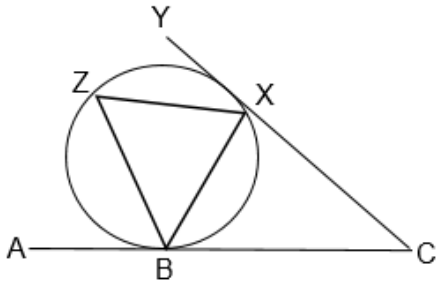


Fig 8. 10

Answers

1. a) BOP = 76°. b) BAP = 38°. c) APB = 90°.
2. FEB, FDB, BED and DFB angles subtended by equal cords.
3. a) XZB = 62.5°. b) BZX = 66°. c) BXZ = 85°.
   d) XCB = 62°. e) ZXY = 82°.

Activity (*8.5*) Circle theorems

Questions

1. Fig 8.21 show a circle ABCY, XYZ is a tangent to a circle at Y. Answer all questions:

   a. Name two angles equal to angle AYZ.

   b. Name two angles equal to angle CXY.

   c. Which angle is equal to angle BYZ?

   d. Which angle is equal to angle BYX?

   e. If angle ACY is 60°, what is angle AYZ?

   f. If angle BYX is 110°, what is angle BAY?

   g. If angle BCY is 130°, calculate angle BAY?

   h. If angle BYZ is 110°, what is angle BCY?

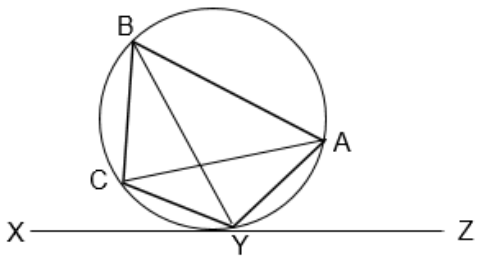


Fig 8.11

Answers

1. a. ABY and ACY b. CBY and CAY c. BCY d. BAY.

   e. 60° f. 110° g. 50° h. 110°.

8.4 Summary

In this unit you have learnt about parts of a circle and circle theorems. You should master them since they are very useful when answering questions on Circle Geometry.

8.5 Further Reading

1. Macrae, M, Madungwe, L and Mutangadura, A (2017) New General Mathematics Book 3, Pearson Capetown.
2. Macral, M, Madungwe, L and Mutangadura, A (2017) New General Mathematics Book 4, Pearson Capetown.
3. Meyers, C; Graham, B; Dawe, L (2004) Mathscape Working Mathematically, 9th Ed, MacMillan
4. Pimentel R and Wall T (2011) International Mathematics, Hodder UK
5. Rayner D (2005) Extended Mathematics. Oxford New York

8.6 Assessment Test

1. Fig 8.22 shows ABC is a circle with centre O. Calculate angle **x**.

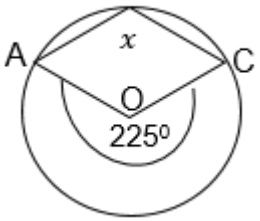


Fig 8. 12

2. The diagram shows AB is a diameter and angle ACD is 35°. Calculate angles:
   a. Angle ABD.
   b. Angle BAD.

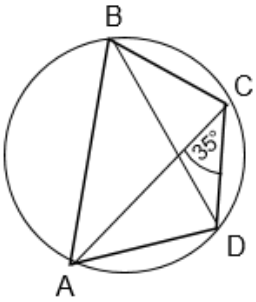


Fig 8. 13

3. In fig 8.25 XRY is a tangent to circle at R, PQ = QR and angle PQR = 70°. Calculate:

   a. Angle PRX. b. Angle QPR. c. Angle QRY.

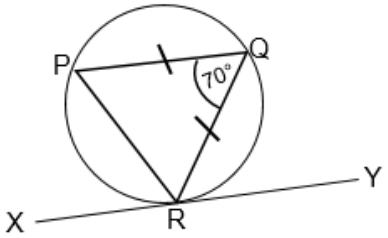


Fig 8. 14

4. Fig 8.26 is such that XR = RZ. Given that RXZ = 50°. Find angle XYZ.

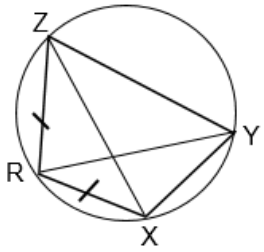


Fig 8. 15

5. Fig 8.27 shows circle centre O, CT and AT are tangent angle AOC = 160°. Calculate:

   a. Angle CAT. b. Angle ABC. c. Angle CTA.

   d. Angle OAC.

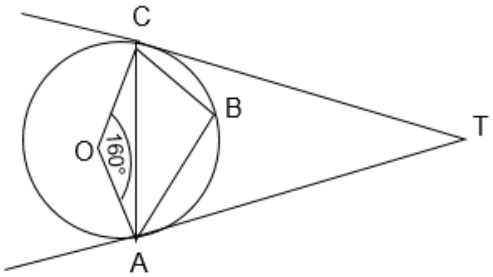


Fig 8. 16

6. In fig 8.28 O is the centre of a circle ABCD, angle ABC = 128°. Calculate sizes of angle marked **x**, **y**, **z** and **r**.

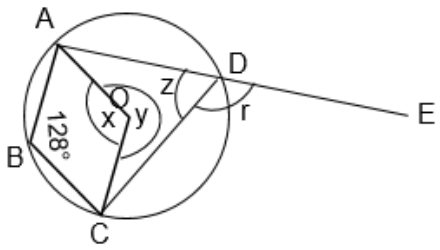


Fig 8. 17

Answers

1. x = 112.5°.
2. a) 35°. b) 55°.
3. a) 70°. b) 55°. c) 55°.
4. 100°.
5. a) 80°. b) 100°. c) 20° d) 10°
6. x = 104°. y = 256°. z = 52°. r = 128°.

UNIT 9 - FINANCIAL MATHEMATICS

CONTENTS

9.1 INTRODUCTION

Everyone handles or deals with money every day, in this unit you are going to learn very important concepts on money. The knowledge you are going to acquire is going to be very useful in your everyday life. You are expected to understand every term which is going to be discussed so that you will be able to make correct calculations.

OBJECTIVES

After going through this unit you should be able to:

- Know what a bank statement is.
- Interpret information found on a bank statement.
- Calculate profit, loss, commission, discount, interests and instalments.
- Solve problems in trading.

KEY WORDS

Interest: - is the money you pay back when you borrow or is the money you get a after saving for a certain period.

Principal: - money borrowed or saved.

Discount: - it is the money deducted from marked price when you pay cash.

Hire purchase (HP): - payment spread over agreed period of time for an item which you cannot pay for cash.

Instalments: - part payment made to cover hire purchase.

Depreciation: - this is reduction in goods value over a period of time.

Compound interest: - non fixed interest paid over a changing principal.

Bank statement: - a bank record of transactions made for the account holder by the bank over, usually a month.

Deposit: - money banked.

Withdrawal: - money taken from the bank.

Commission: - payment given to the worker to encourage hardworking.

TIME

This unit should be covered within 10 hours.

STUDY SKILLS

One of the most important thing to note in the learning of Mathematics is that practice is the key to mastery of all mathematical concepts. You need to solve as many problems in Financial Mathematics as possible for you to grasp the concepts involved in this topic.

9.2 BANK STATEMENT

A Bank statement shows records of all deposits and withdrawals made by the account holder over a period of time during the month of money kept in the bank. It also shows service charges to the bank for keeping the money in their bank. Fig 9.1, shows a bank statement for Mr Hove for the month of April 1 to 30 2018.

Makazvida Bank Chigaramanhenga Branch

Current Account Statement

Mr R Hove

Date: 30 April 2018

10 Makaha Road

DATE	DETAILS:	WITHDRAWALS	DEPOSITS	BALANCE
1 April	Balance brought forward			457.06
2 April	CASH		46.95	504.01
5 April	CASH	200.00		304.01
9 April	CHQ 0047	150.20		153.81
11 April	CHQ 63119		358.15	511.96
13April	Bank Charges	9.25		502.71
15 April	CASH		218.10	720.81
16 April	CHQ 0048	300		420.81
21April	STO Insurance	90.50		330.31
24 April	CHQ 0049	116.20		214.11
27 April	CASH	50		164.11
28 April	SALARY		465.46	629.57
30 April	Bank Charges	15.92		613.65
	Total	**932.07**	**1 088.66**	

Fig 9. 1 – Bank statement

KEY:

CHQ => Cheque

STO => Stop Order

- All the money withdrawn and deposited is entered by dates and at the end of the month it is added to show total amount withdrawn and deposited.
- The balance is calculated at every transaction made and it is carried forward to the next month.
- The bank has deposit forms and withdrawal forms which are used by customers to deposit and withdraw money.
- Customers are charged a certain fee for some transactions. Such charges are known as bank charges.
- Customers with Savings Accounts may receive interest for keeping their money in the bank. Those with Current Accounts are not eligible to getting interest.

Activity (*9.1*) Bank Statement

Questions

Use the bank statement in fig 9.1 to answer the following questions:

1. What was the opening balance for April 2018?
2. What did Mr Hove do to his account on 2 April 2018?
3. What happened to his balance on 2 April 2018?
4. How much was withdrawn on 16 April 2018?
5. What happened to the balance on 16 April 2018?
6. How much did Mr Hove pay as service charges in April 2018?
7. How much money was on cheque number 63119?
8. How did the cheque in number 7 affect the balance on 11 April?
9. What do you understand by deposits and withdrawals?
10. How many deposits were made by Mr Hove in April 2018?
11. If the amount which was withdrawn by cheque number 0048 was used to pay 6 people. How much did each person get if they were paid equal amount?

Answers

1. $ 457.06.
2. Deposit of $ 46.95.
3. Increased $ 504.01.
4. $ 200.00.
5. Reduced to $ 420.81.
6. $ 25.17.
7. $ 358.15.
8. Increased to $ 511.96.
9. Deposit means putting money in the bank.
   Withdrawal means getting money from the bank.
10. 4.
11. $ 50.00.

SOLVING PROBLEMS

Activity 2 will assist you to understand how everyday real-life problem can be solved.

Activity (*9.2*) Problem solving

Questions

Prepare a bank statement for the month of May 2018 using figure 9.1 and the following information.

i. On 3 May 2018 $65, 50 was deposited.
ii. On 7 May 2018 cheque of $100 was written to Mr Gumbo and Mr Hove withdraw $320 for diesel.
iii. On 15 May bank charges for $15 were made by the bank.
iv. On 22 May Mr Hove's salary of $ 465 appeared in the bank.
v. On 24 May Mr Hove got money from the bank to repair his car.
vi. On 25 May the bank cashed cheque number 0050 written by Mr Hove.
vii. On 31 May the bank closed the columns of the bank statement.

Answers

DATE	DETAILS	WITHDRAWALS	DEPOSITS	BALANCE
1 May 2018	BAL/BF			613.65
3 May 2018	Cash		65.50	679.15
7 May 2018	Cheque	100.00		579.15
7 May 2018	Withdraw Cash	320.00		259.15
15 May 2018	Bank Charges	15.00		244.15
22 May 2018	Salary		465.00	709.15
24 May 2018	Cash Withdrawal	500.00		209.15
25 May 2018	Cheque	50.00		159.15
31 May 2018				
TOTAL		985.00	530.00	

9.3 SIMPLE INTEREST

Interest is money added by a bank to the sum deposited by a customer or money charged by a bank to a customer for borrowing from the bank. The money deposited is called **principal**. Interest is calculated on a fixed principal usually on yearly basis.

If interest is calculated for five years and added to the principal, the sum is **total amount**.

The terms below are used when calculating interest:

- Interest (I)
- Principal (P)
- Rate(R) – rate is usually a percentage(%)

- Time (T) – time is usually in years.

The formula for calculating interest is:

$I = \frac{PRT}{100}$

The following example explains the calculations involved for simple interest.

Worked Example [1]

Question

Calculate the simple interest on $500 which is borrowed for 1 year at 10% per annum (p.a)

Solution

$I = \frac{PRT}{100}$

$I = \frac{500 \times 10 \times 1}{100}$

I = $50

Worked Example [2]

Questions

1) Find the simple interest earned on
   a) $350 deposited for 8 years at 4% per annum.
   b) $40 deposited for 3.5 years at 5% per annum.
   c) $150,40 deposited for 2 years 3 months at 12.5%.

Solutions

a) $I = \frac{PRT}{100}$

$= \frac{350 \times 4 \times 8}{100}$

=$112

b) $I = \frac{PRT}{100}$

$= \frac{40 \times 3.5 \times 5}{100}$

= $ 7

c) $I = \frac{PRT}{100}$

$= \frac{150.40 \times 2.25 \times 12.5}{100}$

$= \$\ 42,30$

Worked Example [3]

Questions

How long will it take for a sum of

a) $400 invested at 6% per annum to earn interest of $48

b) $165 invested at 3% per annum to earn interest of $30,40

c) $250 invested at 8% per annum to earn interest of $60,00

Solution

TIP: Make T subject of the equation or formula

a. $I = \frac{PRT}{100}$

$T = \frac{100I}{PR}$

$= \frac{48 \times 100}{400 \times 6}$

$= 2$ years

b. $T = \frac{100I}{PR}$

$= \frac{100 \times 30.40}{165 \times 3}$

$= 6.14$

$= 6$ years

c. $T = \frac{100I}{PR}$

$= \frac{100 \times 66}{250 \times 8}$

$= 3$ years

Worked Example [4]

Question

What rate per year must be paid for a principal of

a) $350 to earn interest of $60 in 2,5 years

b) $160 to earn interest of $45 in 3 years

c) $250 to earn interest of $120 in 7 years

Solution

TIP: Make R subject of the formula

(a) $R = \frac{100 \times I}{PT}$

$= \frac{60 \times 100}{350 \times 2.5}$

$= 6.86\%$

$= 6.9\%$

(b) $R = \frac{100 \times 45}{160 \times 3}$

$= 9.38\%$

$= 9.4\%$

(c) $R = \frac{120 \times 100}{250 \times 7}$

$= 6.86$

$= 6.9\%$

9.4 COMPOUND INTEREST

In Compound Interest, not only is interest paid on the principal amount but interest is paid on the interest. This means it is compounded, that is, it is added to. This may sound complicated but the example below will make it clear.

Interpretation

In calculation there is no T since time is one year thus:

Year 1 => I = PR

Year 2 => I = PR **(This time principal will be principal for year 1 + interest for year 1)**

Year 3 => I = PR **(Principal for year 2 + interest for year 2)**

Worked Example [5]

Questions

a) Calculate compound interest on $300 at 10%per annum for 3 years.

b) Calculate the compound interest earned on $500 at 12.5% for 4 years.

Solutions

a) Year 1 Principal = 300, Rate = 10%

Interest = $300 \times \frac{10}{100}$ = $30

Year 2 Principal = 330, Rate = 10%

Interest = $330 \times \frac{10}{100}$ = $33

Year 3 Principal = 363, Rate = 10%

Interest = $363 \times \frac{10}{100}$ = 36.30

Therefore for the 3 years = 30 + 33 + 36.30 = $99.30

b) Year 1 Principal = 500, Rate = $12\frac{1}{2}$ =12.5

Interest = $500 \times \frac{12.5}{100}$ = $ 62.50

Year 2 principal = (500+62.50), Rate = $12\frac{1}{2}$ =12.5

Interest = $562.50 \times \frac{12.5}{100}$ = $ 70.31

Year 3 principal = (562.50+70.31), Rate = $12\frac{1}{2}$ =12.5

Interest = 632.81 × $\frac{12.5}{100}$ = $ 79.10

Year 4 principal = (632.81+79.10), Rate = $12\frac{1}{2}$ =12.5

Interest = 711.91 × $\frac{12.5}{100}$ = $ 88.99

Therefore total or compound interest = 62.50 + 70.31 + 79.10 + 88.99 = $300.90.

After going through the examples above, you have realised that calculating compound interest is easy. Can you can attempt the following activity?

Activity (*9.3*) Compound interest

Questions

1) Calculate compound interest on:
   a) $4000 at 10% for 2 years.
   b) $800 at 9.5% for 3 years.
2) Peter saves $200 in an account which gives 6.5% per annum compound interest. Calculate:
   a) Total interest for 2 years.
   b) Total amount in the bank at the end of 2 years.
3) What is the time to which $700 will amount in 3 years at 15 % per annum compound interest?
4) What is:
   a) simple interest on $800 at 10% per annum for 3 years?
   b) Compound interest on $800 at 10% per annum for 3 years.

Answers

1. a)

   1st year = 400 × $\frac{10}{100}$ = $ 400

   2nd year = 4400 × $\frac{10}{100}$ = $ 440

   Therefore for 2 years = ($400 + $440) = $840

b)
1st year = $76,
2nd year = $83.22,
3rd year = $91.13
Therefore for 3 years = ($76 + $83.22 + $91.13) = $250.35

2. a)
   1st year = $13,
   2nd year = $13.85
   Therefore for 2 years = ($13 + $13.85) = $26.85

   b) Total amount = $226.85

3. 1st year = $105,
   2nd year = $120.75,
   3rd year = $138.86
   and Total = $1064.61

4. a. Simple interest = $240
   b. Compound interest = $256.80

9.5 COMMISSION

Commission is payment made to a worker relating to the number of sales the worker has made. Commission is paid to encourage people to work hard without being supervised. Commission is often paid to staff in the sales department. The more sales they make the more money they are paid. This encourages the staff to sell as many products as possible. Ice cream vendors, insurance agents and bus drivers are among some of the people who may work on commission. Can you think of others? Commission is calculated as percentage of what one gets through the sale of goods.

Worked Example [6]

Questions

The bus driver gets $500 per month as salary and 5% of the total money she or he cashes in per month, calculate:

a. The commission she or he will receive if she or he cashes in $8 000 per month.
b. Total pay for that month.

Solutions

a. Commission 8000 $\times\frac{5}{100}$ = $400
b. His pay that month $500 + $400 = $900

Worked Example [7]

Questions

If the same driver the following month cashes in $6 000, calculate:

a. Commission.
b. Total take home salary.

Solutions

a. Commission 6000\times\frac{5}{100}$ = $300.
b. His take home salary $500 + $300 = $800.

Worked Example [8]

Questions

If during third month the driver cashes in $10 000, calculate:

a. Commission.
b. That months take home.

Solutions

a. Commission 10000 $\times\frac{5}{100}$ = $500
b. His total salary $500 +$500 = $1000

Supposed the fourth month he decides not to work he will just get $500. Is it good to work for commission? What can you say about that?

Activity (*9.4*) Bank Statement

Questions

1) Chipo sells tickets for watching soccer, if she is paid $1 per every tickets she sells. How much will she get after selling 2 000 tickets as commission?
2) A car sales lady get 2% in a $1. Calculate her commission if she sold $60 525 worth of cars that month. What will be her total salary if her basic salary was $230?
3) A textbook cost $12.50. The author gets 15% per text book sold. In 2017, 10 000 copies of that book were sold calculate the author's commission.
4) A man collects rent at a commission of 6.5%. In June 2018 he collected $12 860. How much was his:
   a. Commission.
   b. If monthly salary was $156, how much money did he take home in June that year?
5) A bank charges 3.5% commission on loans issued out. Calculate the commission the bank get after issuing loans worth $100 000
6) An electrical company worker get 12 cents in every dollar as commission. One month the worker sold 4 stoves at $2 500 each and 2 television sets at $3 150 each. What amount did the worker get at the end of the month as commission? If the worker's basic salary was $350 per month how much altogether did the worker take home?

Answers

1. $ 2000.00.
2. Commission $ 1210.50.
   Total $ 1440.50.
3. $ 18750.00.
4. a) $ 835.90. b) $ 991.90.

5. $ 3500.
6. Commission $ 1956.
   Total $ 2256

9.6 HIRE PURCHASE

Hire purchase allows individuals or businesses to buy fixed assets over a long period of time while making monthly payments which include an interest charge. A person does not have to find a large sum of money to purchase an asset of choice. However, a cash deposit is paid at the start of the period and sometimes interest rates can be quite high which means that buying items through hire purchase is more expensive that paying cash. The interesting thing about hire purchase is that a person is given his/her item/asset of choice as soon as the processes for buying through hire purchased have been initiated and the customer's eligibility to make monthly payments has been verified.

Worked Example [9]

Questions

A new television cost $850 in cash or a deposit of $180 is required and 12 monthly payments of $60. Calculate the hire-purchase price and find the saving when bought through cash.

Solutions

Total Instalments = 12 × $60

= $720

Hire purchase price = deposit ($180) + total instalments ($720)

= $900

Saving when bought through cash is = $900 - $850

= $50

Worked Example [10]

Questions

The price of a car by hire-purchase is $26 500, a 30% deposit is required. The remainder is spread over $1\frac{1}{2}$ years monthly instalments:

a. How much is the deposit?

b. What is the total amount to be paid through the instalments?

c. Calculate the monthly instalment.

Solutions

a. Deposit = $\frac{30}{100} \times \$26\ 500$

= $7 950

b. Total amount to be paid by instalments = $26 500 - $7 950

= $ 18 550

c. Monthly instalment: Time to pay = $1\frac{1}{2}$ = 18 months

Therefore = monthly instalment = $\frac{18\ 550}{18}$ = $103.56

Now that you have gone through the examples of how hire purchase is calculated, can you attempt the following activity.

Activity (*9.5*) Hire purchase

Questions

1) A man can buy a bicycle by either paying $ 930 cash or paying $85 monthly instalment for 13 months:

   a. Calculate the cost of the bicycle through instalments.

   b. What is the difference between cash price and the hire-purchase price?

   c. Express the difference as percentage.

2) A car cost $9 865 cash. Hire-purchase requires a 15% deposit and 36 monthly instalments of $300. How much:

   a. does it cost on hire-purchase.

   b. saved if bought for cash?

3) A refrigerator cost $745 cash, it can be bought by hire-purchase but its price increases by 15%. If 10% is required as deposit and remainder is spread over 12 equal monthly instalments. Calculate the:

   a. Hire-purchase price.

   b. The deposit.

   c. Monthly instalments

TIP: Percentage is calculated on the original amount. Express increase or decrease as a fraction of the original amount then multiply by 100.

Answers

1. a. $1 105 b. $175 c. 18.8%
2. a. $12 279.75

   b. $2 414.75

   a. $856.75 b. $85.68 c. $64.26

9.7 Summary

In this unit we have learnt that: –

- Banks give services to both people and companies.
- The customers are charged for the service they get from the bank.
- The Bank also provide statements at the end of each month to its clients.

We have also learnt about simple interest and compound interest:

- In simple interest the principal is fixed.
- In compound interest the principal changes.
- Hire purchase involves deposits and instalments. Hire purchase is more expensive than buying in cash.

9.8 Further Reading

1. Macrae, M, Madungwe, L and Mutangadura, A (2017) New General Mathematics Book 3, Pearson Capetown.
2. Macral, M, Madungwe, L and Mutangadura, A (2017) New General Mathematics Book 4, Pearson Capetown.
3. Meyers, C; Graham, B; Dawe, L (2004) Mathscape Working Mathematically, 9th Ed, MacMillan
4. Pimentel R and Wall T (2011) International Mathematics, Hodder UK
5. Rayner D (2005) Extended Mathematics. Oxford New York

9.9 Assessment Test 1

1. Cash price of a table is $550 or a deposit of $110 is required the remainder is increased by 15% and paid in 12 monthly instalments, Calculate:
   a. Total amount to be paid by instalments.
   b. Each instalment to be paid monthly.
   c. Percentage increase over cash price.
2. Calculate the simple interest on $600 which is saved for 2 years at 11% per annum. What is the total amount after 2 year?
3. Find the percentage rate per year if simple interest of $160.50 is paid on $4 570.00 invested for 6 month.
4. $500 was left in the bank for how long if it earned $62.00? If the simple interest at a rate of 6% per annum.
5. A bank charges $36 simple interest on a sum of money borrowed for 3 years. If the rate of interest is 12% per annum, find the sum of money.
6. Calculate the compound interest earned on $850, at $12\frac{1}{2}$ % for 5 years.

Answers for Assessment Test 1

1. a. $506. b. $42.17. c. $12%.
2. $132, $732.
3. 7%.
4. 2 years.
5. $100.
6. $681.33.

Assessment Test 1

1. Mrs Dude borrowed $9 000 from the bank. She is expected to pay it back after 9 months. If she paid back $10 060. Calculate the annual simple interest rate charged by the bank.
2. Zodwa got a loan for $6 500, at 7% per annum, for 3 years. Calculate the amount she paid back if the interest earned was calculated on:
   i. Simple interest.
   ii. Compound interest.
3. Use the extract of a bank statement in table 9.2 to calculate:
   i. The values of x, y, z and w.
   ii. Total deposits.
   iii. The closing balance.

DATE	DETAILS	DEBIT	CREDIT	BALANCE
1/6/19	Balance Brought Forward			3495
11/6/19	Cash Deposit		120	3615
21/6/19	Withdrawal	x		3105
25/6/19	Cheque Withdrawal	250		y
30/6/19	Bank Charges	12		z
30/6/19	Salary		815	w

Table 9. 1

4. A circle centre O and points A, B, C, D, E and F lie on the circle. BF is the diameter of the circle, as shown in fig 9.2. find the following angles:
   i. Angle FAB.
   ii. Angle FAE.
   iii. Angle CAB.

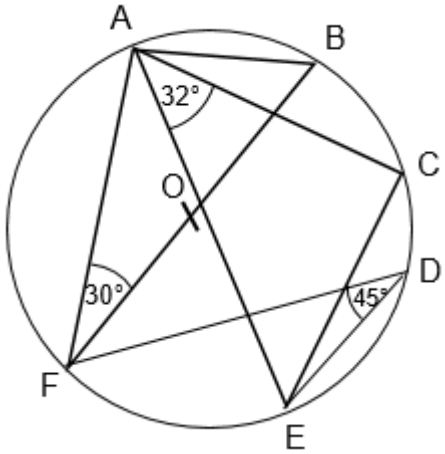


Fig 9. 2

5. O is the centre of a circle XCD. AXB is a tangent, angle XCD = 80° and ODC = 30°, as shown in fig 9.3. Find the values of x, y and z.

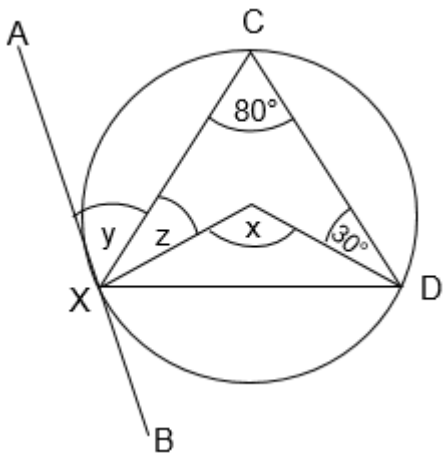


Fig 9. 3

6. ABCD is a cyclic quad in which ADC = 78°, BAC = 18° and CBD = 42°, as shown in fig 9.4. Calculate:
   a) The angles of triangle BAD.
   b) What name is given to triangle BAD?

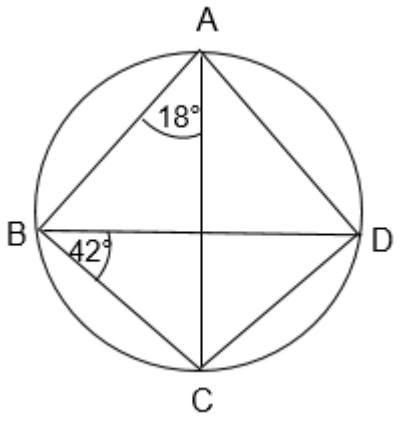


Fig 9. 4

7. The angles of a quadrilateral taken in order are x, 5x, 4x and 2x. calculate:
   a) The sizes of the angles.
   b) What is the quadrilateral called?
8. The sum of 3 angles of a nonagon sides is 462°, the remaining other angles are all equal.
   a) How many sides has a nonagon?
   b) Calculate the size of each angles.

Answers for Assessment Test - Paper 2

1. 15.7%.
2. i. $7865. ii. $7 962.
3. i. x = $110, y = $ 2 855, z = $2 843 and w = $1 3058. ii. $120.
4. i. 90°. ii. 45°. iii. 13°.
5. x = 160°, y = 40° and z = 50°.
6. a. 60°, 60° and 60°. b. Equilateral.
7. a. 30°, 150°, 120° and 60°. b. Trapezium.
8. a. 9 sides. b. 133°.

UNIT 10 GEOMETRY 3

CONTENTS

10.1 INTRODUCTION

This unit will dwell much on the similarity ratios of areas and volumes. This concept is a derivative of similarity and congruency covered at Level 1. Various examples will be used to simplify the concept for you. The assessment section will also contain typical examination question papers. You are also expected to recall the calculation of volumes of solid shapes.

OBJECTIVES

After going through this unit, you should be able to

- calculate the scale factors
- area of plane and solid shapes
- volume factors of plane and solid shapes

KEY TERMS

Similar: – objects are similar if they show equivalency in characteristics except for size.

Equiangular:– refers to similar triangles, equal in angle sizes, but different in size of the shapes.

TIME: You are expected not to spend more than 8 hours on this unit.

STUDY SKILLS

The key skill to mastery of mathematical concepts is practice. You need to solve as many Geometry related problems as possible for you to grasp all the concepts in this topic.

10.2 SIMILARITY

Two polygons are said to be similar if

(a) they are equi-angular and

(b) the corresponding sides are in proportion.

For triangles, being equi-angular implies that corresponding sides are in proportion. The converse is also true.

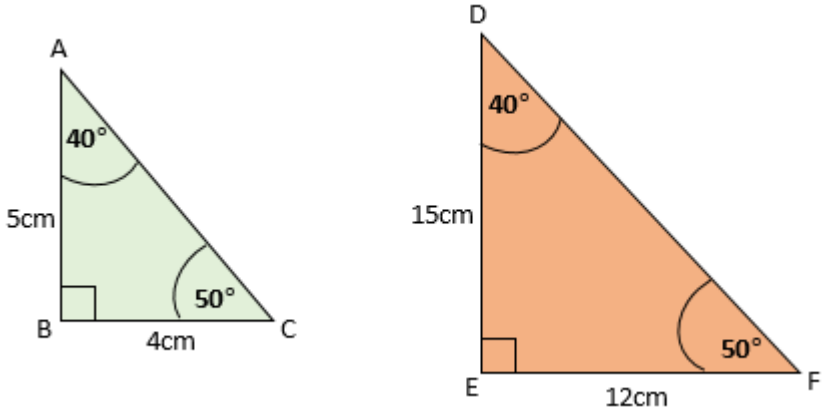
10.3 SCALE FACTOR

This concept was well covered at Level 1 and the example below will serve as a reminder of the concept and this will assist you to understand the area and volume factors quite easily.

Worked Example [1]

Questions

In the diagrams given below are two similar shapes. Calculate the scale factor.



Solutions

The triangles given above are equiangular that means they are similar and their scale factor can be found using the formula stated below.

Scale factor = ratio of the corresponding sides

Scale factor $= \frac{\mathrm{EF}}{\mathrm{BC}} = \frac{15}{5} = 3$

Or

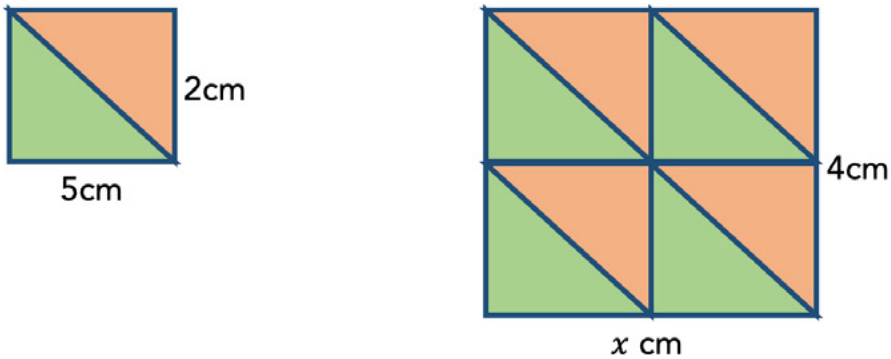
Scale factor $= \frac{\mathbf{DE}}{\mathbf{AB}} = \frac{\mathbf{12}}{\mathbf{4}} = \mathbf{3}$

10.4 THE AREA FACTOR

The ratio of the areas of two similar shapes or figures is the square of the scale factor of the two shapes.

Worked Example [2]

Questions



The diagram above are two similar shapes. Calculate

(a) Scale factor (b) The value of x (c) The area factor

Solution

(a) Scale factor $= \frac{4}{2} = 2$

(b) $\frac{x}{5} = \frac{4}{2}$

$$x = 5 \times \frac{4}{2} = 10\text{cm}$$

Or

$$x = 5 \times \text{scalefactor} = 5 \times 2 = 10\text{cm}$$

(c) The area factor $= \frac{\text{areaofbigrectangle}}{\text{areaofsmallrectangle}} = \frac{4\times10}{2\times5} = \frac{40\text{cm}^2}{10\text{cm}^2} = 4$

Or

The area factor $= (\text{scale})^2 = 2^2 = 4$

Worked Example [2]

Questions

Area **A** has an area of 12m^2 and a similar area **B** has an area of 1.5m^2. If area **A** has one of its dimension equal to 310 cm. Find the corresponding dimension of area **B.**

Solution

Area factor of area **B** to area **A =** $\frac{1.5}{12} = \frac{1}{8}$

Scale factor = square root of area factor = $\sqrt{\frac{1}{8}}$

Length of dimension A = 310cm

Length of dimension B = scale factor × dimension

$$= \sqrt{\frac{1}{8}} \times 310$$

$$= \frac{1 \times 310}{\sqrt{8}}$$

$$= 109.6 \quad (3.\text{s.f})$$

NOTE : Examiners are also interested in scale drawing and map interpretation, therefore we need to do an example of that nature.

Worked Example [4]

Question

A map is drawn to a scale of 1: 25500. Find the area in hectares of an irrigation field which has an area of 20 cm^2. **Hint: 1 ha = 10 000m^2**

Solution

Scale factor = 25 500

Area factor = $(25\ 500)^2 = 650\ 250\ 000$

Irrigation field = $650\ 250\ 000 \times 20\text{cm}^2$

$$= \frac{650\ 250\ 000 \times 20}{100 \times 100}\text{m}^2$$

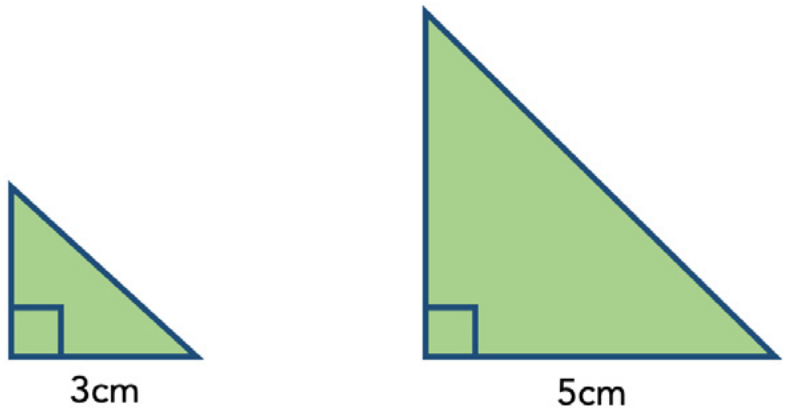
$$= \frac{650\ 250\ 000 \times 20}{10000 \times 100 \times 100}\ \text{ha}$$

$$= 130.05\ \text{ha}$$

Activity (*10.1*) Area of similar shapes

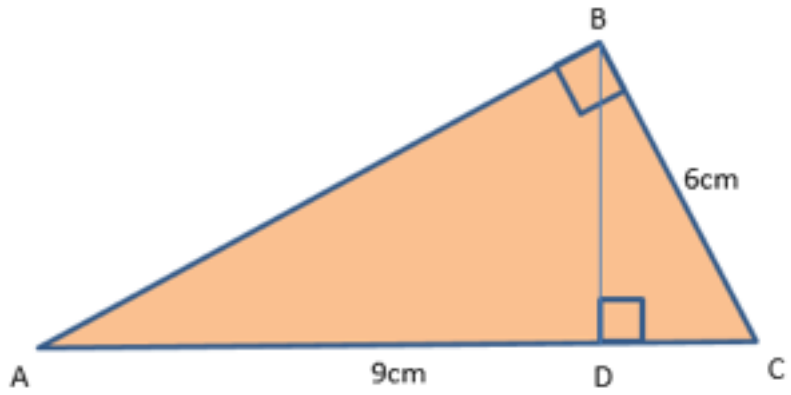
Questions

1)



In the diagram above are two similar triangles with corresponding sides in the ratio 5:3. Find the ratio of their areas

2)



In the diagram above triangles ABC and BDC are right angled.

(a) State in correct order the two triangles similar to triangle DCB

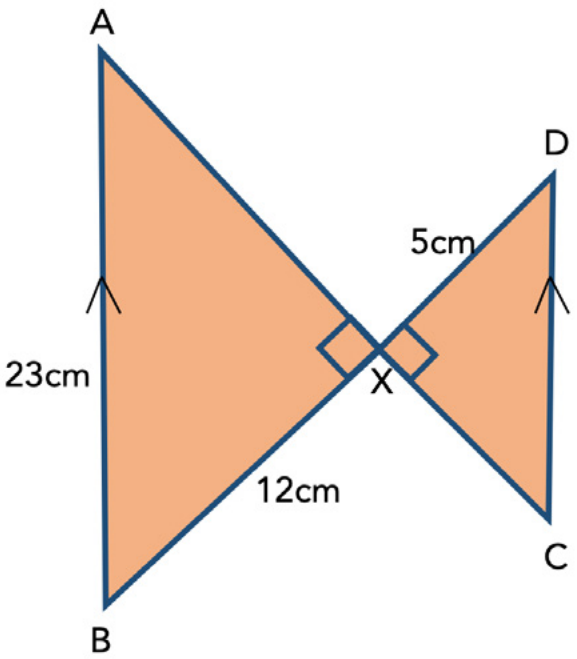
(b) Find

(i) side DC

(ii) scale factor of triangle ABC and BDC

(iii) the area factor of triangles stated in (ii)

3)



In the diagram above shows two right angled triangles where straight lines BD and AC intersect at X. Calculate

(a) The areas of the two triangles

(b) The area factor of the larger triangle to the smaller one.

4) Two pentagons have corresponding sides of 3cm and 7cm. Find

(a) the ratio of their areas

(b) area of the smaller pentagon, if the area of the larger one is 350 m^2

(c) area of the larger pentagon, if the area of the smaller one is 133m^2

5) The ratio of the area of two circles is ¾. Find

(a) the ratio of their radii.

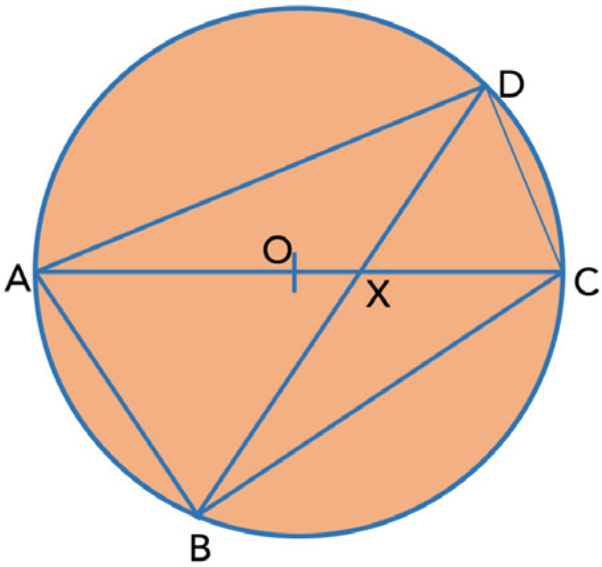
(b) the radius of the smaller circle, if the radius of the larger one is 24cm

(c) the radius of the larger circle, if the radius of the smaller one is 15 cm

6) A cultivated land covers an area of 8 hectares. Find the area in cm^2 of the cultivated land when drawn on a map of scale 1: 50 000

7) The area of the window pane of a sliding door is $3m^2$. In a reduced photograph of the house, the window pane is a rectangle 10cm by 2cm. Find the length and breadth of the real window pane

8)



(a) Write down the triangle that is similar to triangle AXB

(b) If $\frac{AX}{DX} = \frac{3}{2}$, find the ratio $\frac{\text{Area of triangle BAX}}{\text{Area of triangle DCX}}$

Answers

1. 25:9

2. (a) ΔDBA and ΔBCA

(b) (i) DC = 3 (ii) 2:1 (iii) 4:1

3. (a) 20.4 and 117.7 (b) 144:25

4. (a) 9:49 (b) 64.3 (c) 124.1

5. (a) $\sqrt{3} : 2$ (b) $12\sqrt{3}$ (c) $10\sqrt{3}$

6. 0.32 cm^2

7. $2\sqrt{1500}$ cm × $10\sqrt{1500}$

10.5 THE VOLUME FACTOR

The ratio of the volumes of the similar solids is the cube of the scale factor of the two solids

The diagrams below are two similar solid cylinders, one of height $6h$ and radius $3r$, the other of height $3h$ and radius r

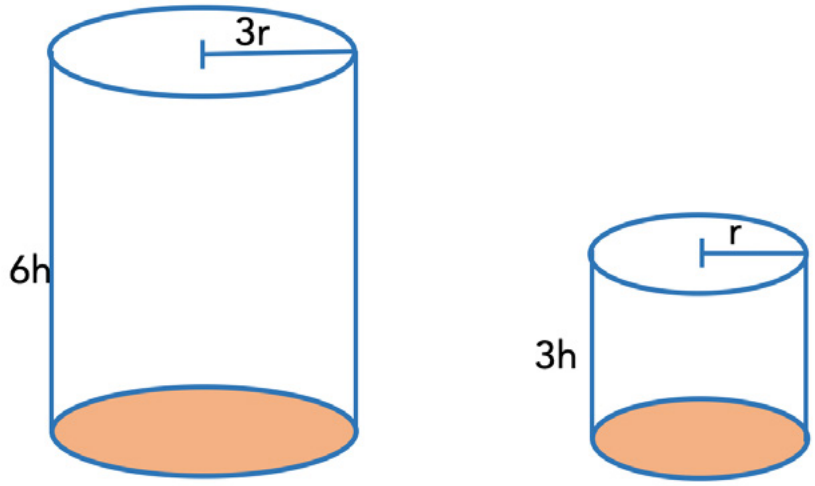


Fig 10.1 – The Volume Factor

The heights and radii are of the same ratio and it shows that the shapes are similar.

The scale factor of the small cylinder to the large cylinder $= \frac{2h}{6h}$ or $\frac{r}{3r} = \frac{1}{3}$

<u>Remember</u> (Volume of a cylinder $=\pi r^2 h$ **)**

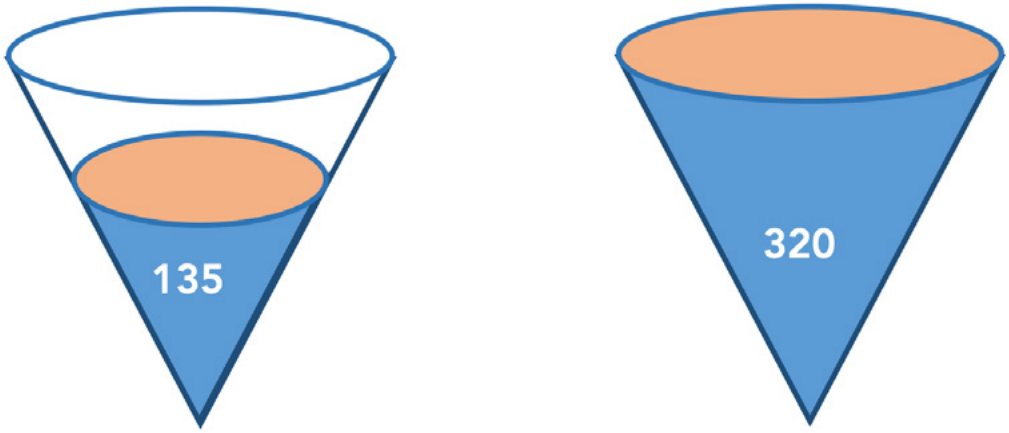
Volume factor of the small cylinder to the big cylinder

= ratio of volumes

$$= \frac{\pi r^2(2h)}{\pi(3r)^2(6h)} = \frac{2}{9\times 6} = \frac{1}{27} = \left(\frac{1}{3}\right)^3$$

Worked Example [5]

Questions



Two similar conical containers shown above contain 135 ml and 320ml of cough syrup respectively. If the vertical height of 135ml of syrup is 6cm, find the vertical height of 320ml of syrup.

Solution

Capacity is proportional to volume, thus, ratio of volumes = ratio of capacity

$$= \frac{135}{320} = \frac{27}{64} = \frac{3^3}{4^3} = \left(\frac{3}{4}\right)^3$$

Scale factor = $\frac{3}{4}$

Height of 135ml = 6cm

Height of 320ml = $\frac{4}{3} \times 6 = 8\text{cm}$

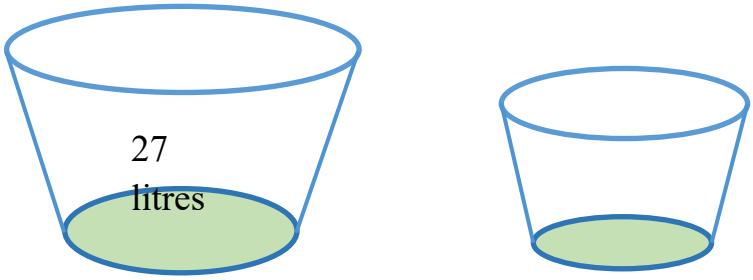
Activity (*10.2*) Volumes of similar shapes

Questions

1) Two similar tanks are 42m and 28m high respectively. If the larger tank holds 12 000 litres of water. Find the capacity of the smaller one.
2) An architecture designs a scale model of a building. The volumes of air in the scale model and the actual building are 30 000m^3 and 220m^3. If the height of the window in the actual building is 4.8m, find the height of the door in the scale model.

3) An ice-cream cone costs RTGS $2. How much would a similar ice-cream cone, 4 times the height and diameter cost?

4)



The diagram above shows two similar frustums with capacities of 26 litres and 8 litres respectively. Find

(a) the ratio of heights

(b) the height of the larger frustum if the smaller one is 36cm high

5) Two circular holes are similar in shape and have diameters of 10cm and 30cm. If water is poured into the holes, find

(a) the ratio of their heights

(b) the capacity in litres of the larger hole if the smaller hole holds 2litres when full.

6) Two bolts are similar in shape and have radii of 1,2mm and 4.8mm

(a) Find the ratio of their masses

(b) If the mass of the larger bolt is 30g, find the mass of the larger bolt.

Answers

1. 3555.6

2. 0.933

3. (a) 3:2 (b) 121.5

4. (a) 1:3 (b) 54

5. (a) $\frac{1}{64}$ (b) 1920

REFLECTION

- scale factor is the basis of area factor and volume factor
- knowledge of plane and solid shapes is vital in similar shapes
- similarity and congruency covered at level one is important in this unit

10.6 SUMMARY

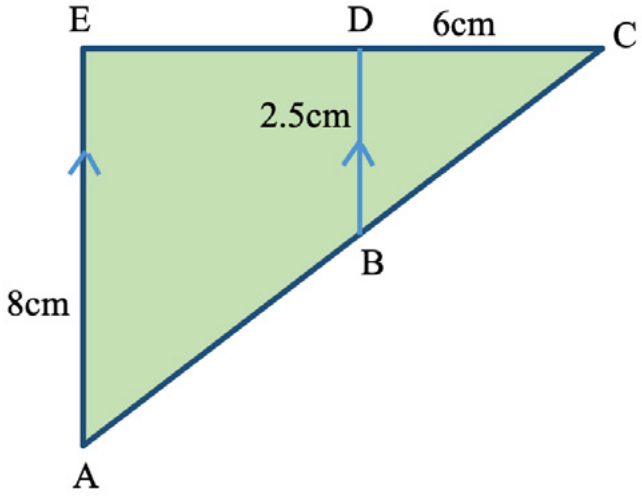
In conclusion, we have made it simple for you to understand the concept of area factors and volume factors by revisiting the aspect of similarity in some of our examples. Various examples have been included to cater for examination questions in the assessment section. Try to refer to the given examples where you encounter advanced questions in the assessment section.

10.7 Further Reading

1. Macrae, M, Madungwe, L and Mutangadura, A (2017) New General Mathematics Book 3, Pearson Capetown.
2. Macral, M, Madungwe, L and Mutangadura, A (2017) New General Mathematics Book 4, Pearson Capetown.
3. Meyers, C; Graham, B; Dawe, L (2004) Mathscape Working Mathematically, 9th Ed, MacMillan
4. Pimentel R and Wall T (2011) International Mathematics, Hodder UK
5. Rayner D (2005) Extended Mathematics. Oxford New York

10.8 Assessment Test

1.



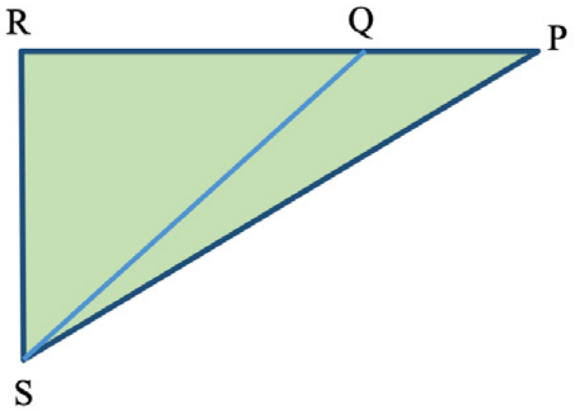
Given that AE = 8cm, BD = 2.5, CE = 6cm and area of $\Delta BCD = 24cm^2$.

(a) State two similar triangles

(b) Find the scale factor and area factor of the two similar triangles

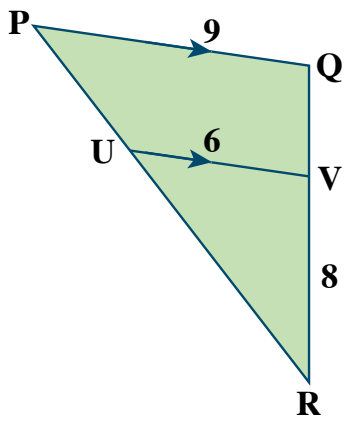
(c) Find the area of the trapezium ABDE

2. **ZIMSEC JUNE 2003 P2 QUESTION 6(a)**



In the diagram PQ:PR = 2:5. Given that the area of triangle QRS is 15cm2, find the area of triangle PRS

3. ZIMSEC JUNE 2004 P2 QUESTION 12(b)



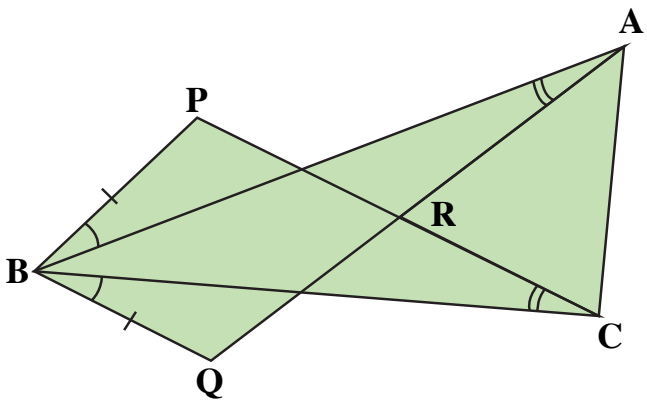
In the diagram, PQ is parallel to UV, PQ = 9cm, UV= 6cm and VR = 8cm.

(i) Name, in correct order the triangle which is similar to triangle VUR.

(ii) Calculate VQ.

(iii)Given that the area of triangle VUR = 32cm^2, calculate the area of the trapezium PUVQ. [7]

4. ZIMSEC NOV 2000 P2 QUESTION 4(b)

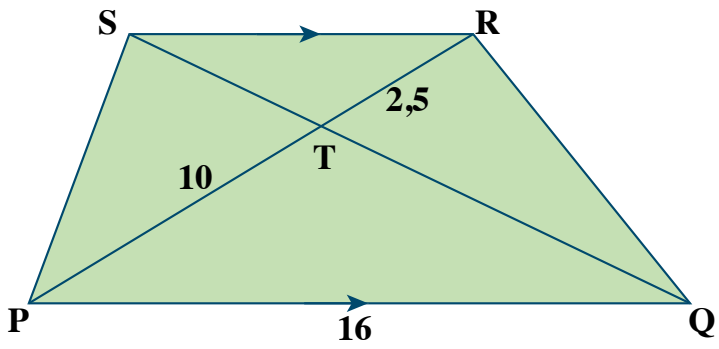


In the diagram, BP= BQ, P$\hat{B}$A = C$\hat{B}$Q and B$\hat{A}$Q = B$\hat{C}$P.

(i) Name, in correct order, the triangle that is congruent to triangle ABQ. State the case of congruency.

(ii) Hence write down pairs of equal elements, not already given, in the congruent triangles. [5]

5. ZIMSEC NOV 2002 P2 QUESTION 4(b)



In the diagram, SR is parallel to PQ, and SQ and PR intersect at T.

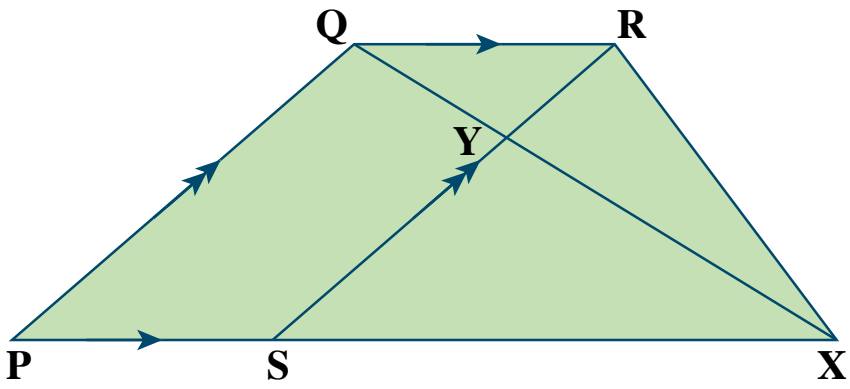
Given that PT = 10cm, RT = 2,5cm, and PQ = 16cm, calculate

(i) SR,

(ii) the ratio of the area of triangle RTS to the area of triangle PTQ,

(iii) the ratio of the area of triangle PTQ to the area of triangle QTR. [6]

6. ZIMSEC NOV 1994 P2 QUESTION 5(b)



In the diagram, PQRS is a parallelogram. X is a point on PS produced such that PX= 3PS and the area of the parallelogram is 12 cm^2.

(i) Find the area of triangle QRX. [2]

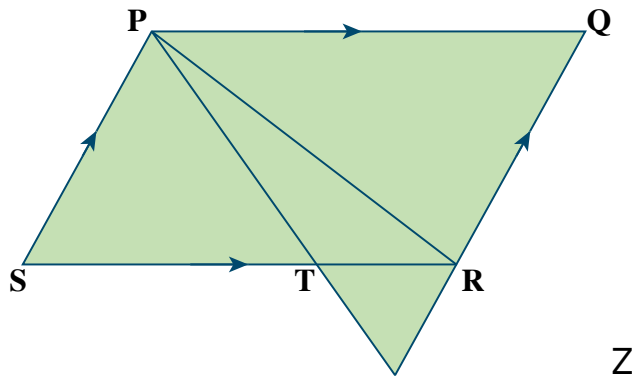
(ii) Calculate the following ratios giving your answers as common fractions in their lowest terms:

(a) $\frac{QR}{SX}$, [1]

(b) $\frac{\text{the area of } \triangle\ QRY}{\text{the area of } \triangle\ XSY}$, [2]

(c) $\frac{\text{the area of } \triangle\ XYR}{\text{the area of } \triangle\ XYS}$. [2]

7. ZIMSEC NOV 1994 P2 QUESTION 5(b)



In the diagram $PQRS$ is a parallelogram.
QR is produced to V so that $QR = 3RV$ and PV cuts SR at T.

(a) Name, in the correct order, two triangles that are similar to triangle VRT. [2]

(b) Write down the ratio of

(i) $\frac{VT}{VP}$, [1]

(ii) $\frac{VR}{PS}$. [2]

(c) Taking the area of triangle VRT to be 4 cm^2, calculate the area of

(i) triangle PRT, [2]

(ii) triangle VPQ, [2]

(iii) parallelogram $PQRS$. [3]

8. ZIMSEC NOV 2005 P2 QUESTION 9(b)

Two circles have areas of $6\frac{1}{4}$ m^2 and $2\frac{1}{2}$ m^2 respectively.

(i) Express the area of the smaller circles as a percentage of the area of the large circle.

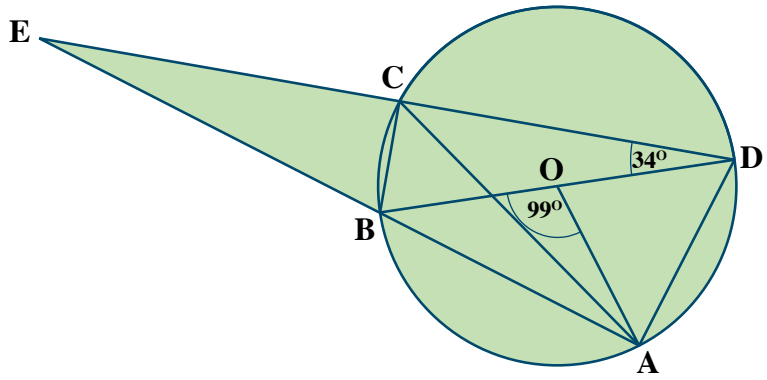
(ii) Calculate the ratio $\frac{\text{radius of the larger circle}}{\text{radius of the smaller circle}}$

leaving your answer in surd form.

(iii) (In this part **take** π **to be** $\frac{22}{7}$).

Calculate the radius of the larger circle. [7]

9.



M is the midpoint of CD.

(a) Explain why triangle OMD is similar to triangle BCD.

(b) Write down the value of $\dfrac{\text{Area of } \triangle OMD}{\text{Area of } \triangle BCD}$. [5]

10. CAMBRIDGE SYLLABUS D 2005 MAY/JUNE QN 7

[The volume of a pyramid is $\frac{1}{3} \times$ base area $\times$ height.]

[The volume of a sphere is $\frac{4}{3}\pi r^3$.]

Morph made several different objects from modelling clay.
He used 500 cm^3 of clay for each object.

(a) He made a square-based cuboid of height 2 cm.

Calculate the length of a side of the square. [2]

(b) He made a pyramid with a base area of 150 cm^2.

Calculate the height of the pyramid. [2]

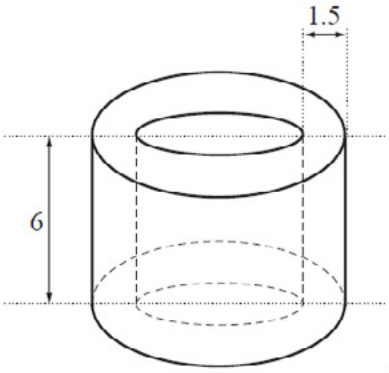
(c) He made a sphere.

Calculate the radius of the sphere. [2]

(d) He wrapped the clay around the curved surface of a hollow cylinder of height 6 cm.

The thickness of the clay was 1.5 cm.

Calculate the radius of the hollow cylinder.



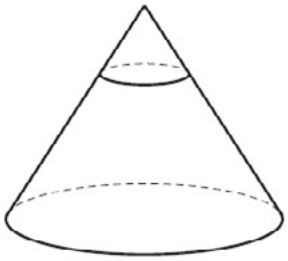
[4]

(e) He made a cone.

Then he cut through the cone, parallel to its base, to obtain a small cone and a frustum.

The height of the small cone was two-fifths of the height of the full cone.

Use a property of the volumes of similar objects to calculate the volume of clay in the small cone.

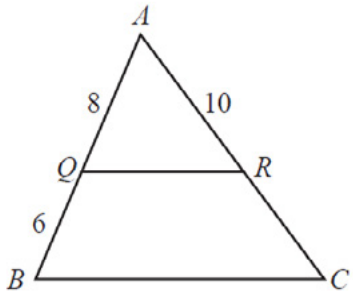


[2]

11. CAMBRIDGE SYLLABUS D 2006 MAY/JUNE QN 6(b)

In the diagram, triangle AQR is similar to triangle ABC.

$AQ = 8$ cm, $QB = 6$ cm and $AR = 10$ cm.



(i) Calculate the length of RC. [2]

(ii) Given that the area of triangle AQR is 32 cm^2, calculate the area of triangle ABC. [2]

12. CAMBRIDGE SYLLABUS D 2007 MAY/JUNE QN 7

A, B, C, D and E are five different shaped blocks of ice stored in a refrigerated room.

(a) At 11 p.m. on Monday the cooling system failed, and the blocks started to melt.
At the end of each 24 hour period, the volume of each block was 12% less than its volume at the start of that period.

(i) Block A had a volume of 7500 cm^3 at 11 p.m. on Monday.

Calculate its volume at 11 p.m. on Wednesday. [2]

(ii) Block B had a volume of 6490 cm^3 at 11 p.m. on Tuesday.

Calculate its volume at 11 p.m. on the previous day. [2]

(iii) **Showing your working clearly**, find on which day the volume of Block C was half its volume at 11 p.m. on Monday. [2]

(b) **[The volume of a sphere is $\frac{4}{3}\pi r^3$.]**
[The surface area of a sphere is $4\pi r^2$.]

At 11 p.m. on Monday Block D was a **hemisphere** with radius 18 cm.

Calculate

(i) its volume, [2]

(ii) its **total** surface area. [2]

(c) As Block E melted, its shape was always **geometrically similar** to its original shape.
It had a volume of 5000 cm^3 when its height was 12 cm.

Calculate its height when its volume was 1080 cm^3. [2]

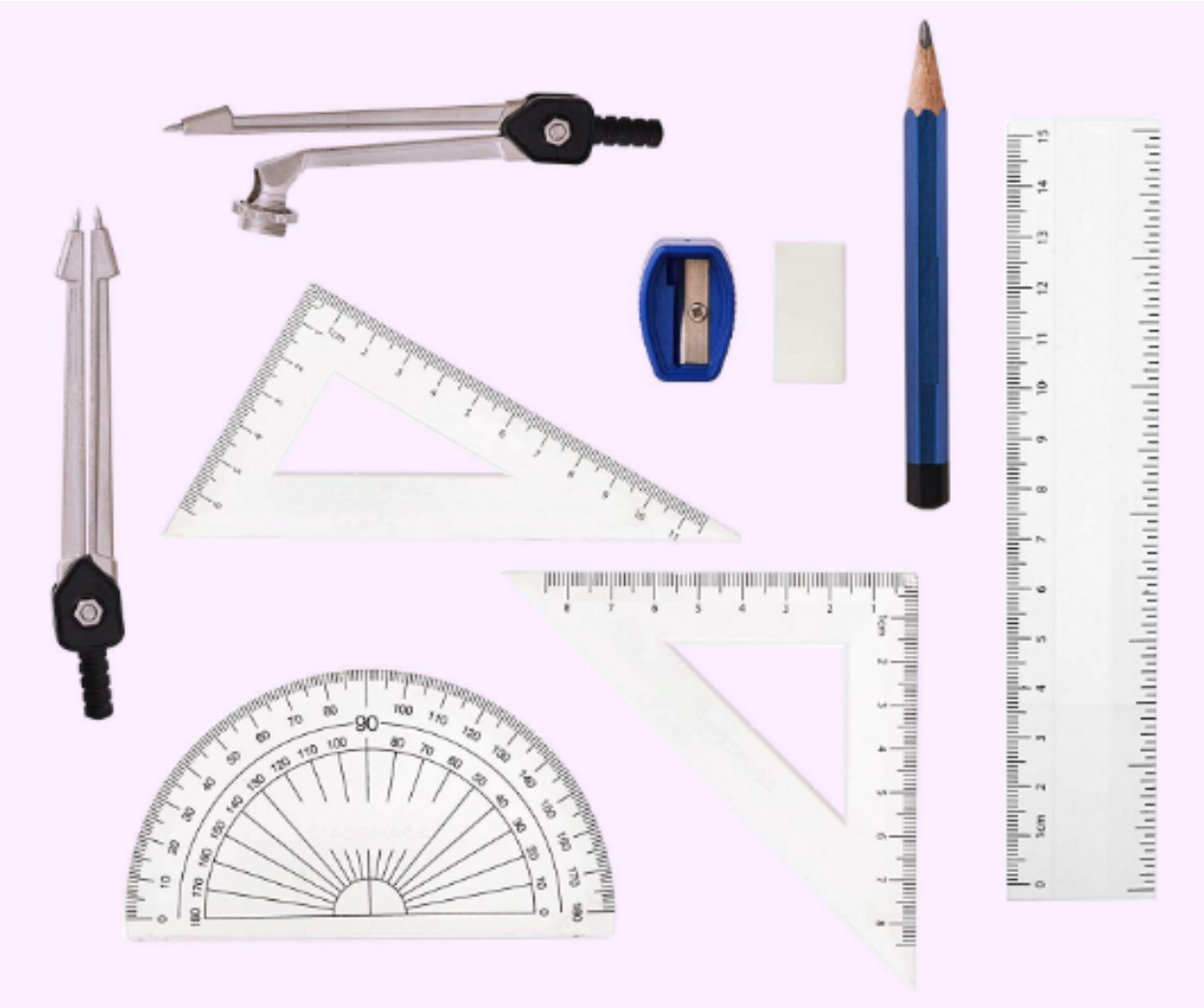
UNIT 11 - GEOMETRY 4 (Geometrical Construction)

CONTENTS

11.1 INTRODUCTION

Have you ever wondered how bridges, buildings and cars are perfectly and accurately designed? How exactly did man succeeded in the journey to the moon?

Yes you are wondering and thinking about it. Well, all this is made possible by simple mathematical construction which is the main thrust of this unit. In the unit you will be introduced in the use of geometrical instruments which you will find in a mathematical set. You will learn how to construct lines, angles and plane shapes. You would also learn the laws of locus. The picture below shows what the unit is all about



OBJECTIVES

After going through the unit, you should be able to:

- ✓ construct straight lines
- ✓ bisect straight lines
- ✓ construct angles (60°, 120°, 90°)

- ✓ bisect angles (to make 30°, 45°, 150° and 135°)
- ✓ construct triangles
    - given all the sides
    - give two sides and an angle
    - given two angles and one side

KEY WORDS

The following are key words and their meaning in mathematics

Construction - is the drawing of shapes, angles and lines accurately using geometrical or mathematical instruments
Bisection - this is to divide into two equal pieces or parts
Locus - is a set of points satisfying a given rule
Vertex - the point of intersection of any lines on a given sharp. It might be at times called corners of a shape.
Equidistant - the word is made up of two words, equal and distance therefore the word means being at an equal distant from the same point or thing.
Supplementary angles – these are two angles which add to 180°.

TIME

You need to be done with this unit in 10 hours. Within the ten hours, two hours should be the contact time with your tutor, airing out challenging areas.

STUDY SKILLS

Make sure that your geometrical instruments are tightened and the pencil is sharpened for accuracy. To be successful in construction you must practice the skills as often as you can. You must not move to the next concept before grasping the prior (previous topic). To be sure of what you are doing, you can use a protractor to measure your angles.

11.2 CONSTRUCTING A STRAIGHT LINE

Before construction takes place, all lines have names. A straight line from point A to point B is called line AB. Can you name a line from C to D? YES indeed it is called line CD.

Here is your first example on construction of lines below

Worked Example [1]

Questions

Construct a straight line AB which is 8 cm long.

Solution

Step1 - Draw a long unmeasured straight line. Select a point on the line which is not at the immediate starting end of the line as shown below and label the point A

Fig 11.1 – Drawing a straight line

Step 2 - Open a radius of 8cm. this is done by placing your compass on the ruler. The needle end point aligned to zero and the pencil aligned to 8.

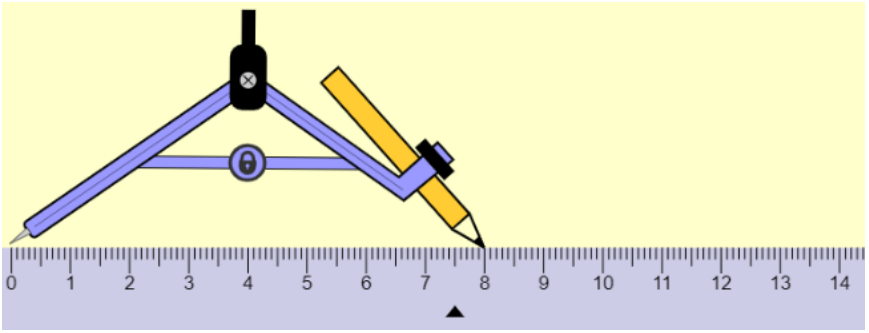


Fig 11.2

Step 3 - Without changing the radius on the compass place the needle end on A on your line and the pencil end point will make an arc on the line and label the point B.

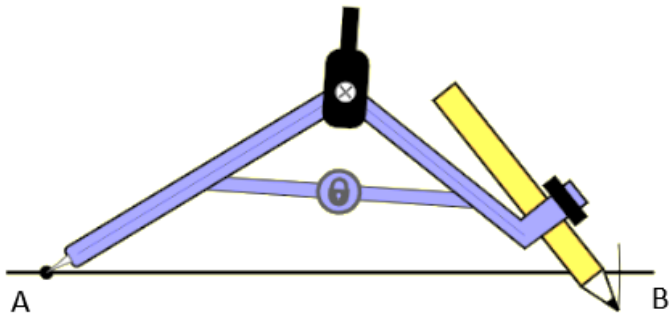


Fig 11.3

Step 4 - Without changing the radius on the compass, with the needle end point at B, make an arc to cross at A.

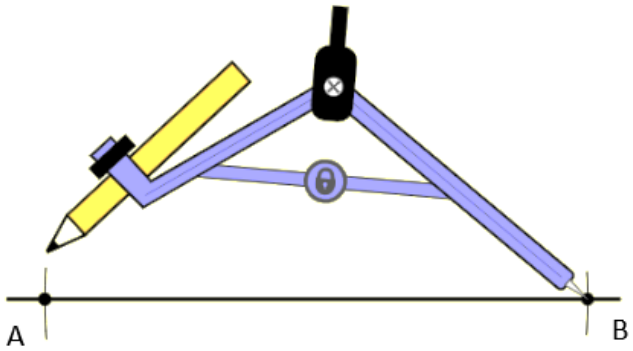


Fig 11.4

Activity (*11.1*) Construction of a line

Questions

1. Construct a line AB=9cm.
2. Construct a straight line CD=6.5cm long.
3. Construct line XY=4.7 cm.

11.3 BISECTION OF STRAIGHT LINES

What is to bisect? Yes! it is to divide into two equal parts or pieces.

Worked Example [2]

Questions

Bisect line AB=8cm long

Solution

Step 1 - Open your compass to a reasonable size (not too large or too small).With the needle end at A, make the arc above and below line AB,

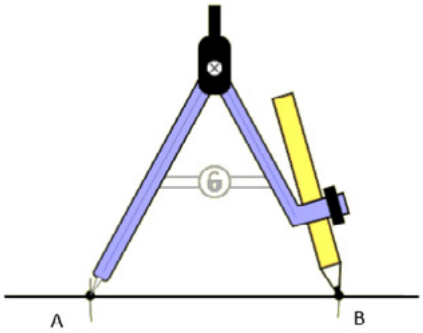


Fig 11.5

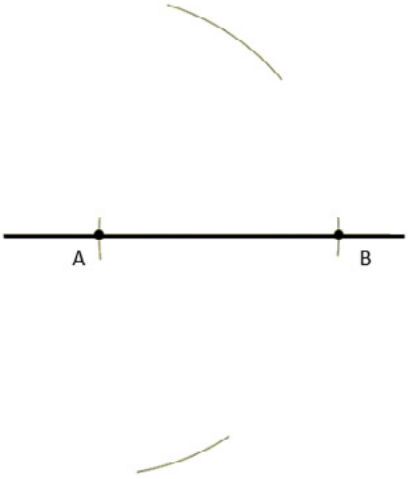


Fig 11.6

Step 2 - without changing the radius on the compass, with the needle end now at B, make arcs above and below AB. The arcs INTERSECT

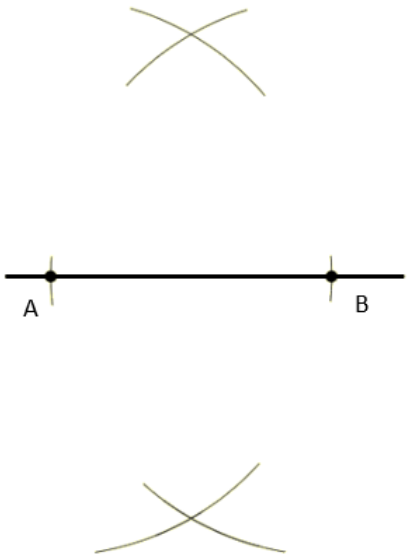


Fig 11.7

Step 3 - Where the arcs intersect, join with a ruler, thus making a line bisector

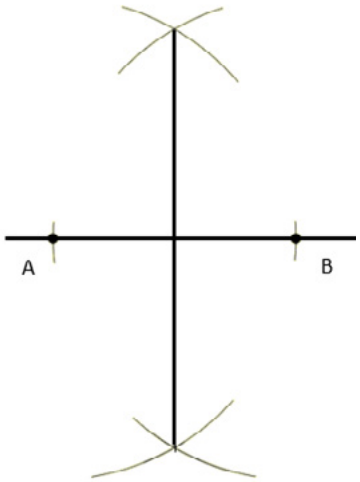


Fig 11.8

Activity (*11.2*) Bisection of a straight line

Questions

1. (a) Bisect the line AB=9cm

   (b) Measure the length from A to the transverse line.

2. (a) Bisect line CD=6,5cm

   (b) Measure the length from C to the transverse line.

3. (a) Bisect line XY=4,7cm

   (b) Measure the length from Y to the transverse line.

11.4 CONSTRUCTION OF ANGLES

You are going to construct a 60° angle, 120° angle and a 90° angle.

11.4.1 Construction of a 60° angle

The following are steps in the construction of a 60°

Step 1 - To construct a 60° angle, we first draw a line of any length. Mark two points on the line and label them A and B.

Fig 11.9

Step 2 - Open the compass to the same dimensions as the two points on your original line.

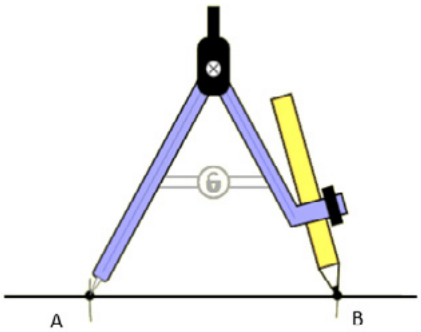


Fig 11.10

Step 3 - Place the point of the compass on point A and draw an arc. Repeat this at the other end (at point B) and the arcs should intersect.

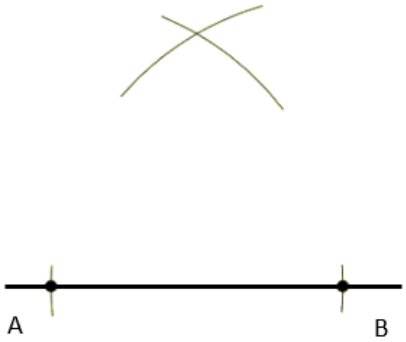


Fig 11.11

Step 4 - Draw a line connecting the tip of the triangle to point A and you will have a 60° angle.

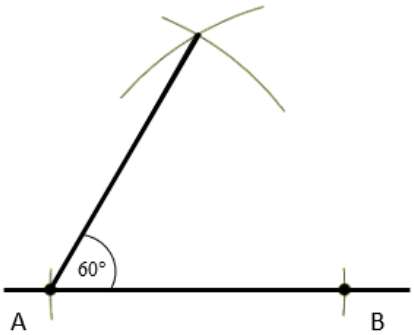


Fig 11.12

If all is done correctly, there should be a 60° angle and a 120° angle. Do you notice how you get the 120° angle.

Points to note;

The two angles add to 180° (angles on a straight line). When a 60°angle is constructed, simultaneously one makes a 120° angle.

11.4.2 Construction of a 120° angle

The following are steps in the construction of a 120° angle

Step 1 - Draw an unmeasured straight line and mark a point and label the point X

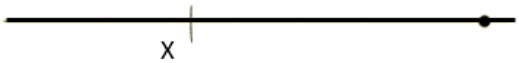


Fig 11.13

Step 2 - Open a radius of your choice (of course not too large or too small) make a long arc as shown with the needle end point at the chosen point X.

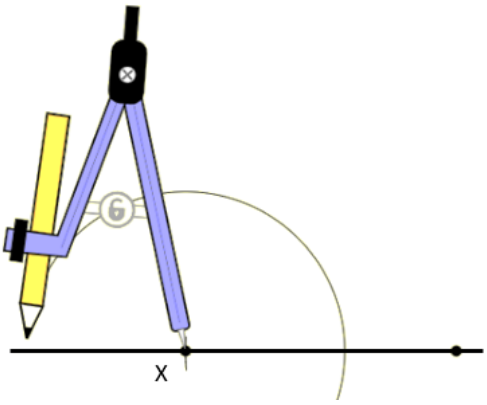


Fig 11.14

Step 3 - Without changing the radius on your compass, with the needle end point at the point of intersection of the line and the long arc, make an arc to cross the longer arc.

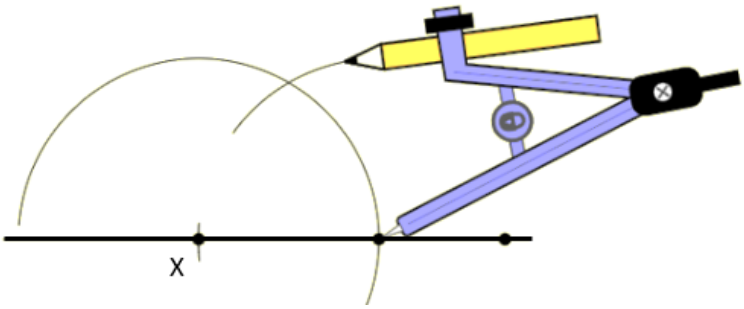


Fig 11.15

Step 4 - Without changing the radius, the needle end point touches the point of intersection of the. Two arcs and make another arc as shown on the diagram

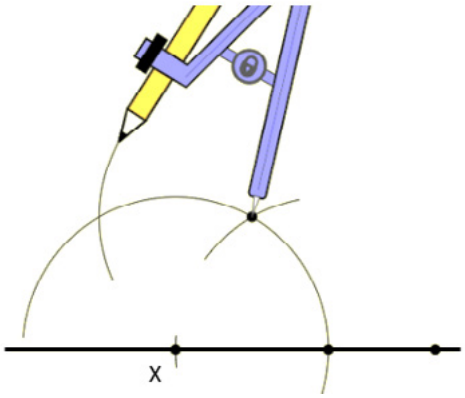


Fig 11.15

Step 5 - Join with a straight line the last constructed arc with the chosen point (point X)

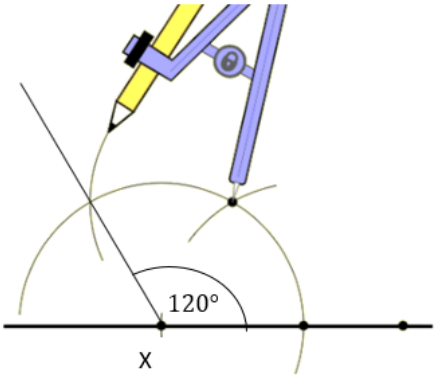


Fig 11.17

Do this yourself;

a. Measure the angle to the right using your protractor……°

b. Measure the angle to the left using your protractor…… °

Observation

……

……

Think about it; if all is done correctly, there should be a 120° angle and a 60° angle

Points to note; The two angles add to 180° (angles on a straight line)

When a 120°angle is constructed, simultaneously one makes a 60° angle.

11.4.3 Construction of a 90° angle.

The following are steps to construction of a 90° angle

Step 1 - Construct an unmeasured straight line

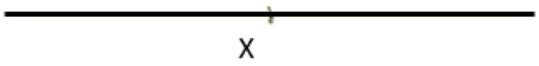


Fig 11.18

Step 2 - Open a radius of your choice on the compass, with the needle end pointed or touching at point X. Construct an arc to the left and to the right of point X as shown.

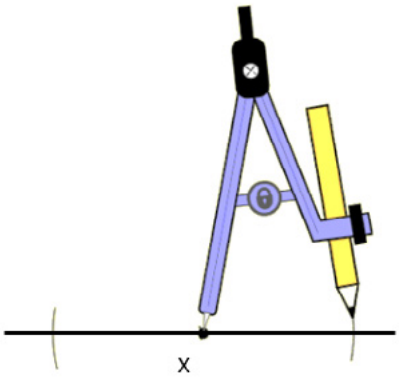


Fig 11.19

Step 3 - Adjusting your radius a bit (increasing the radius increases accuracy and avoids congested work) with the needle point at the right arc, make an arc above the line and below the line, as shown in the diagram.

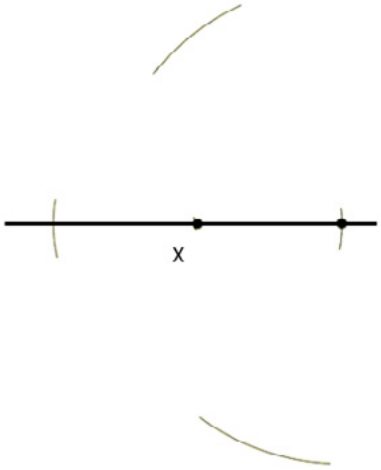


Fig 11.20

Step 4 - Without changing the radius of your compass, place the needle end at the other arc (the right arc) make an arc above and below the line and the arcs intersect as shown

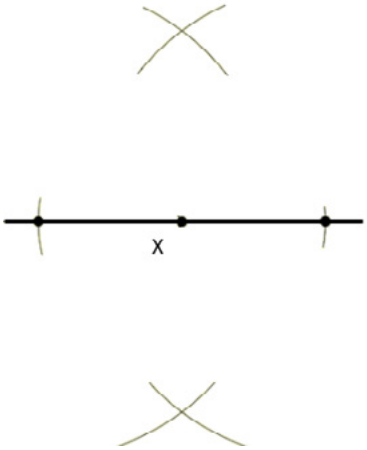


Fig 11.21

Step 5 - Join the point of intersection with a straight line.

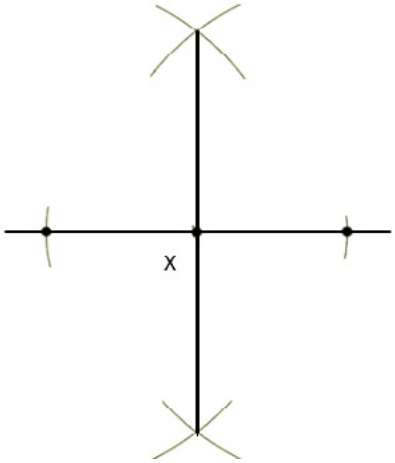


Fig 11.22

Do this yourself;

a. Using the protractor, measure the angles marked A, B, C and D as shown in the diagram below. (measure the angles on your diagram you have constructed)

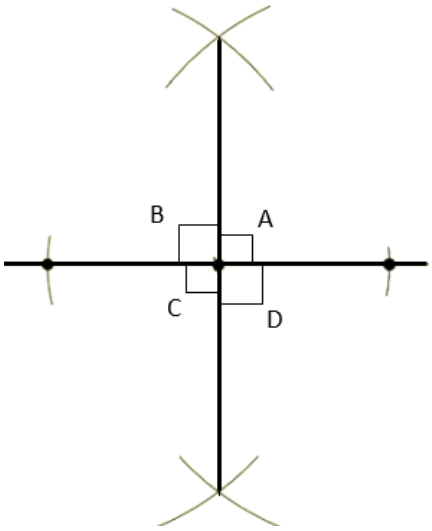


Fig 11.23

Activity (*11.3*) Constructing angles

Questions

Using ruler and compass only

1) Construct a 60° angle
2) Construct a 120° angle
3) Construct a 90°angle

11.5 BISECTION OF ANGLES

Looking back; What is to bisect? Yes! it is dividing into two equal pieces.

Here are the steps in bisecting an angle

Step 1 - Construct an angle to be bisected

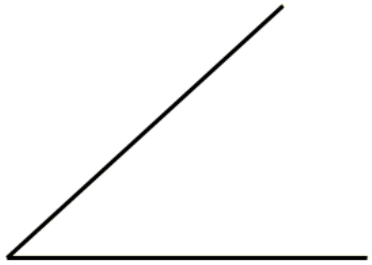


Fig 11.24

Step 2 - Open a radius of your choice and make an arc to cross the two lines making up the angle

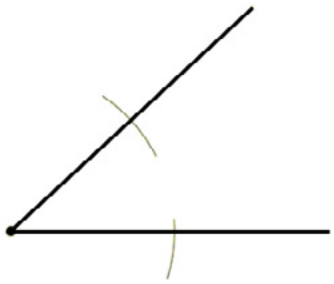


Fig 11.25

Step 3 - Adjusting your radius (making it large increase accuracy and avoids clumps work) with the needle end at one of the intersection point of the arc and the line, make an arc as shown.

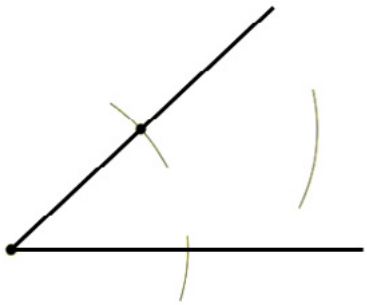


Fig 11.26

Step 4 - Changing the point where the needle end touches, make another arc similar to the one on stage three above, the two arcs will cross as follows.

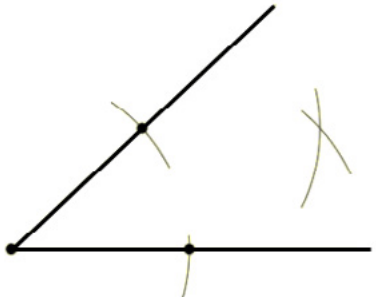


Fig 11.27

Step 5 - Join the intersecting point of the arcs with a straight line to where the angle is

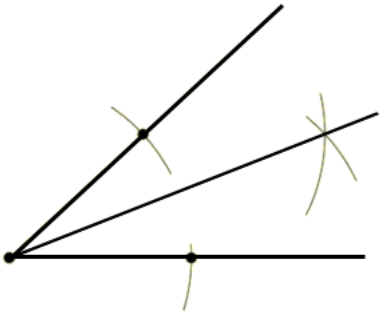


Fig 11.28

Bisection of a 90° angle

TIP; to bisect an angle you first have to construct the angle in Question

Worked Example [3]

Questions

Bisect a 90° angle

Solution

Step1 - Construct a 90° angle

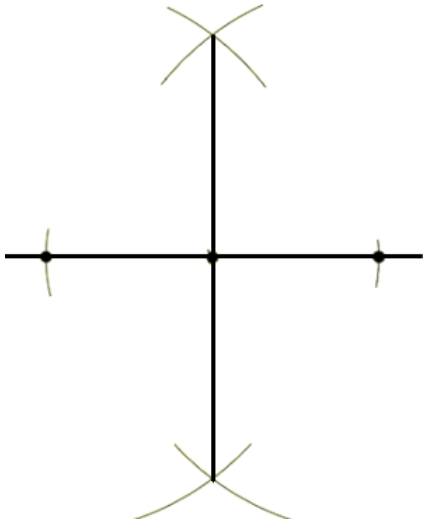


Fig 11.29

Step 2 - Open a radius of your choice on your compass with the needle point at the 90° angle, make arcs to cut through the lines making the 90°angle.

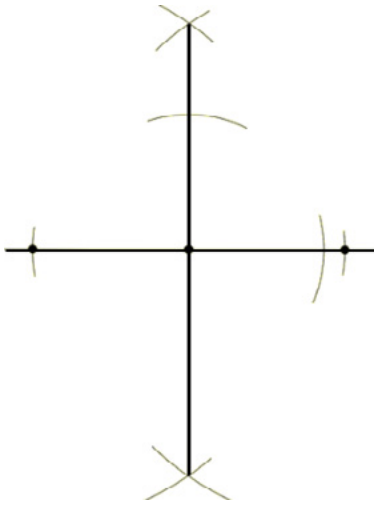


Fig 11.30

Step 3 - Adjusting your radius, make arcs to intersect as shown

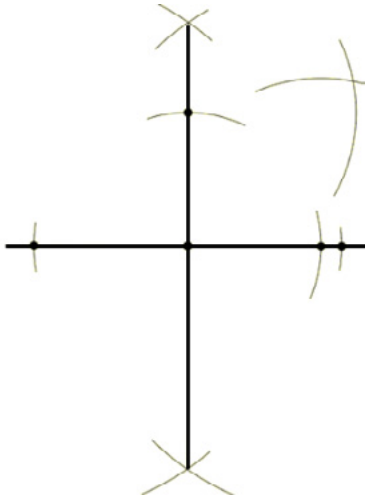


Fig 11.31

Step 4 - Join the point of intersection of the arcs with a straight line to the angle

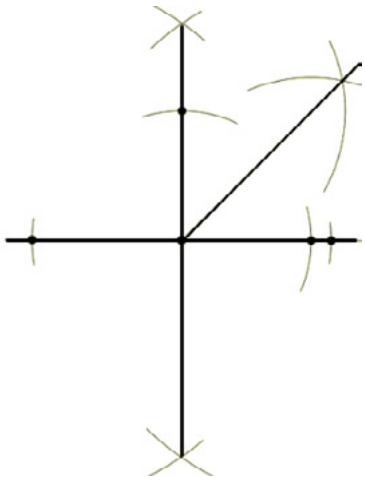


Fig 11.32

Do this yourself;

a. Measure the angle made after bisection to the left of the diagram
b. Measure the angle created to the right of the diagram

What did you observe?

.....

.....

.....

If all this is done correctly, 45° and 135° angles are created. And note that the two angles add to 180°

Worked Example [4]

Question

Bisect a 60° angle

Solution

Step 1 - Construct a 60 °angle

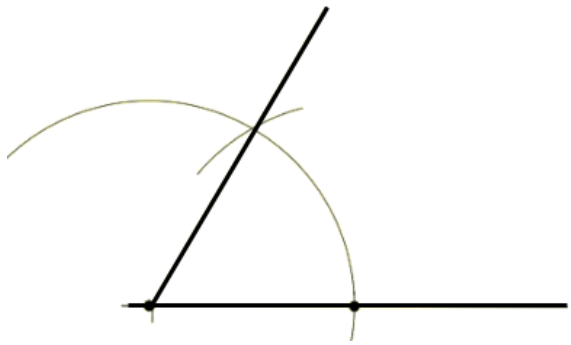


Fig 11.33

Step 2 - Open a radius of your choice. Not too small or too big, make arcs to cross the two lines making the 60°angle

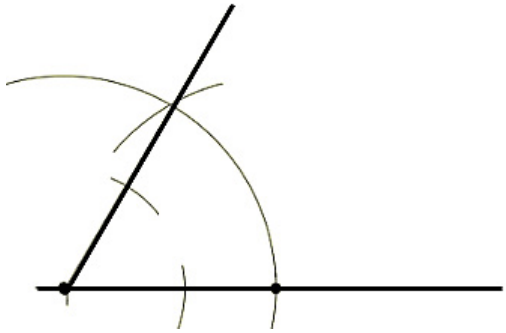


Fig 11.34

Step 3 - With the needle point on the arcs, to make intersecting arcs

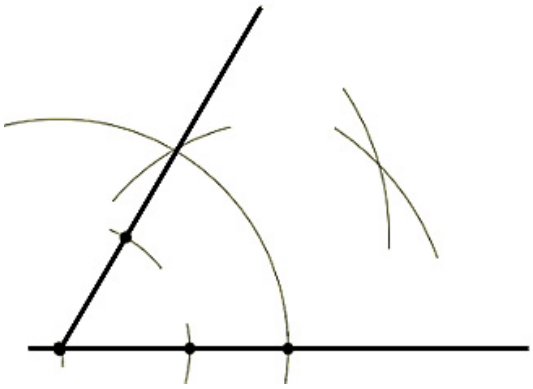


Fig 11.34

Step 4 - Join the intersection point of the arcs with a line to the vertex with the angle (60°)

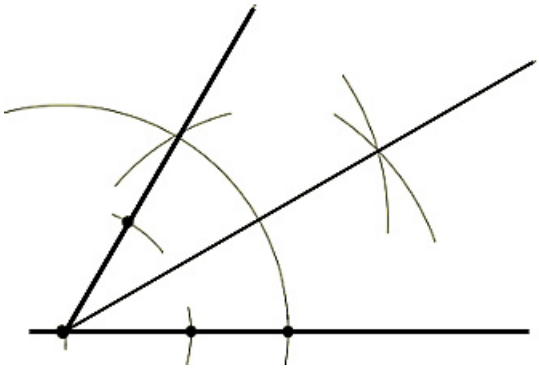


Fig 11.36

Do this yourself

a. Measure the angle to the left of your diagram……°

b. Measure the angle to the right of the diagram……..°

What did you observe?

.....

.....

.....

If all this is done correctly you would make a 30°angle and a 150° angle

TIP; if the Question requires you to construct a 150° angle, just construct a 30° angle and if a 135° angle is required, just construct a 45 °angle.

Activity (*11.4*) Construction and bisection of angles

Questions

Using a ruler and compass only, construct on separate diagrams

a. 30°
b. 135°
c. 150°
d. 45°

11.6 CONSTRUCTION OF PLANE SHAPES

Here we are going to construct the following plane shapes; triangles, parallelograms, trapeziums, rhombus and kites.

11.6.1 Construction of Triangles

 # Given all the 3 sides of the triangle

Worked Example [5]

Question

Construct a triangle ABC with AB=5cm, BC=4cm and AC= 5cm

Solution

Step 1 - Construct line AB=5cm (by now you are capable of constructing a line) labeling clearly A and B

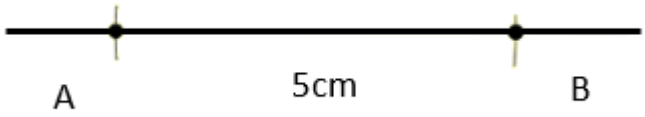


Fig 11.37

Step 2 - Open a radius of 4 cm to make line BC, with the needle point touching at point B, make an arc above line AB

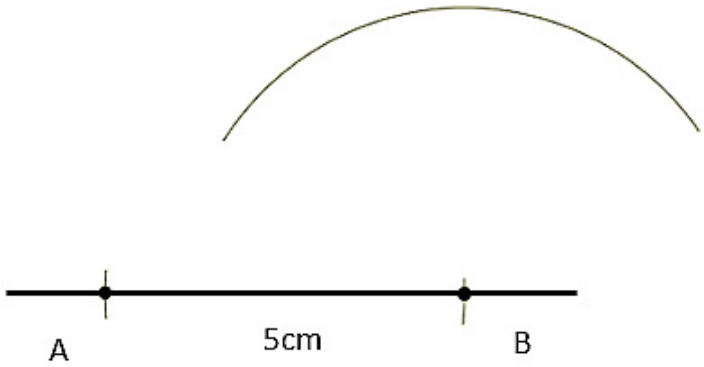


Fig 11.38

Step 3 - Open a radius of 7 cm to make the line AC, with the needle point at A, Make an arc above line AB to intersect another arc and label the point of intersection point C

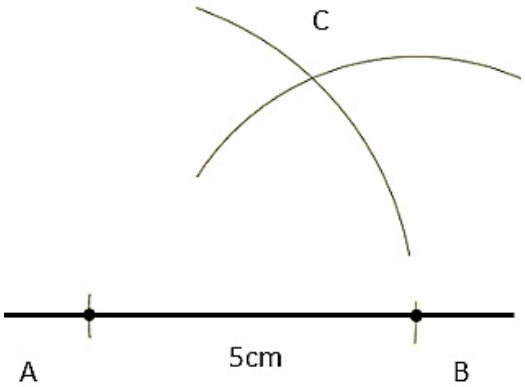


Fig 11.39

Step 4 - Join point C with a straight line to point A and to point B to make a complete triangle.

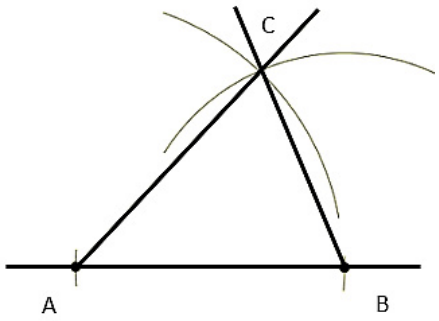


Fig 11.40

Do this yourself

a. Measure the length BC……..cm
b. Measure the length AC……..cm

If all this done correctly, BC=6cm and AC= 7cm

Activity (*11.5*) Construction of angles

Question

Using ruler and compass only, show all construction lines and arcs

1. Construct triangle ABC with AB=8cm, BC=5cm and AC=4cm
2. Construct ΔXYZ with XY=8.5cm, YZ=7cm and XZ=5,3cm
3. Construct ΔDEF with DE=4cm, EF=5 cm and DF=7 cm

 # Given two sides and angle

Worked Example [6]

Question

Construct ΔABC with AB= 5cm, ABC =90° and BC = 6cm

Solution

Step 1 - Construct line AB=5cm

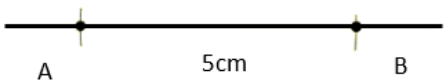


Fig 11.41

Step 2 - Construct a 90° angle at point B.

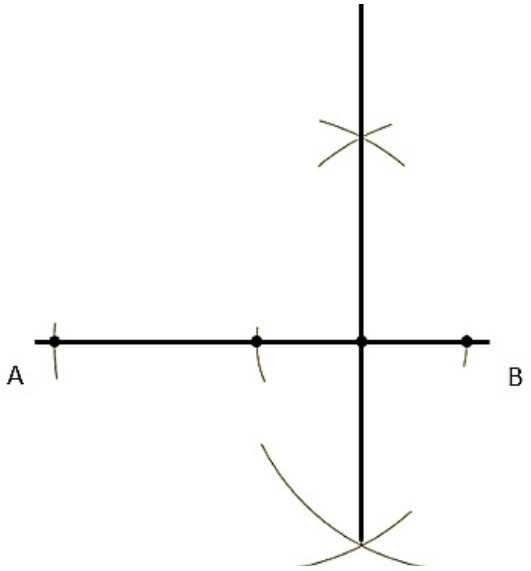


Fig 11.41

Step 3 - Open a radius of 6 cm on your compass and the needle point being at B, make an arc on the line making the 90° angle and label the point C

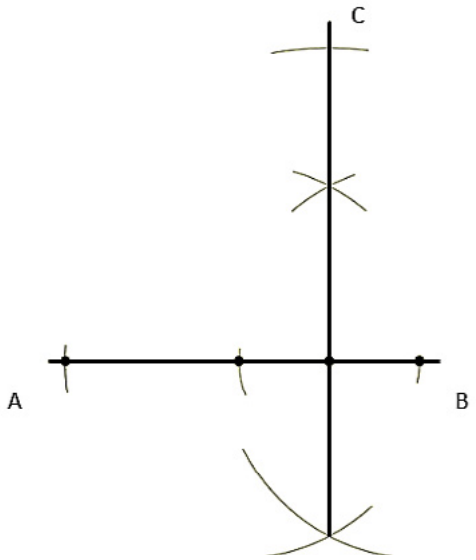


Fig 11.43

Step 4 - Join the point A and C with a straight line to make the triangle ABC.

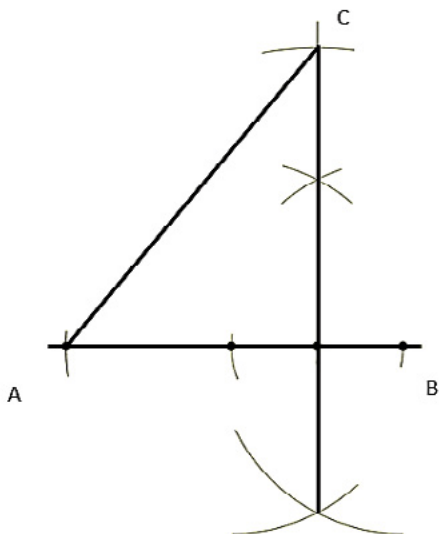


Fig 11.44

Do this yourself

a. Measure using a protractor

i. BAC=……°

ii. BCA=……°

If all this have been done correctly , the angles inside the triangle should add up to 180^0.

Worked Example [7]

Question

Construct triangle XYZ with XY=5cm, angle ZXY =120^0 and YZ=6cm.

Solution

Step 1 - Construct line XY=5cm

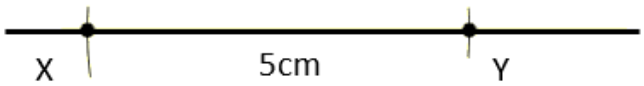


Fig 11.45

Step 2 - Construct a 120^0 angle at point X.

TIP: Alternatively –construct a 60^0 thus using the concept of supplementary angles of a straight line. That is angles in a straight line add up to 180^0

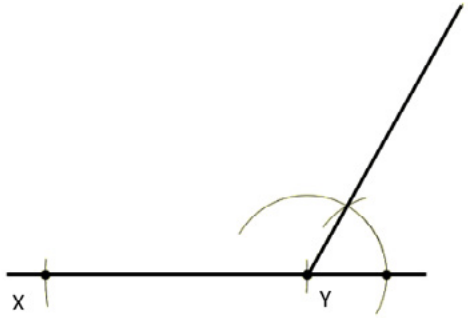


Fig 11.46

Step 3 - Open a radius of 8cm on your compass to construct line XZ. With the needle point on X, make an arc on the line with 120^0 and mark it Z as shown below

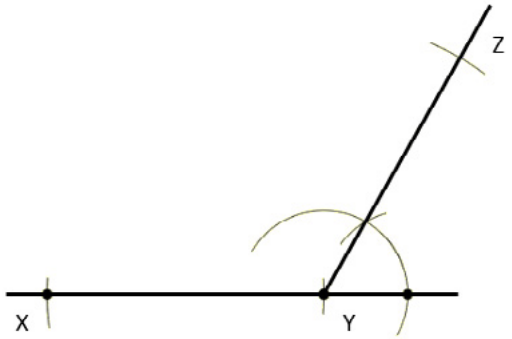


Fig 11.47

Step 4 - Join the point Z with Y with a straight line to make triangle XYZ

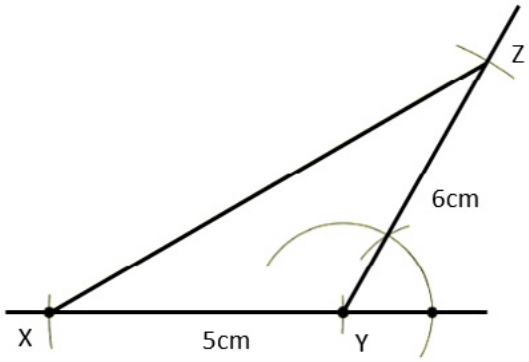


Fig 11.48

Do this yourself,

Calculate the length of XZ in the space below

Activity (*11.5*) Construction of plane shapes

Questions

Using a ruler and compass only

1. Construct triangle ABC with AB =7cm, angle ABC =120^0 and BC =4 cm .
2. Construct triangle XYZ with XY = 9.5 cm and angle YXZ =135^0 and XZ=7cm

 # Given two angles and a side

Worked Example [8]

Question

Construct a ΔABC with AB=5cm, ABC=90° and BAC=30°

Solution

Step 1 - Open a radius of 5cm and construct a line AB=5cm and on point B, construct a 90°angle

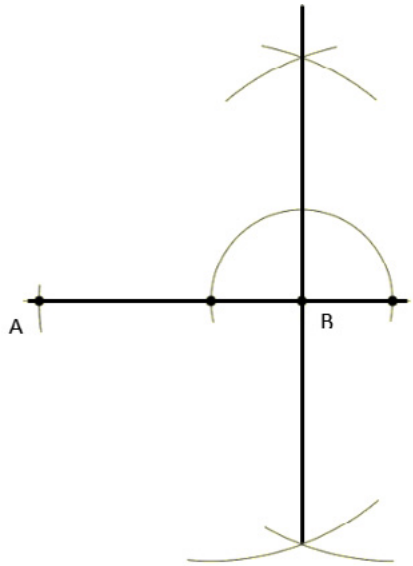


Fig 11.49

Step 2 - Construct an angle of 60° at point A, then bisect the angle to make the required angle of 30°. Label the point of intersection of the line C.

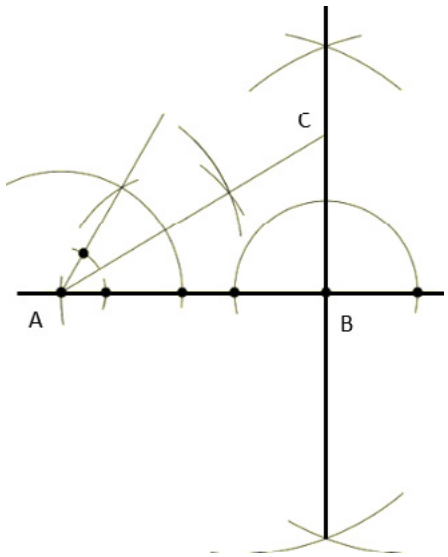


Fig 11.50

Worked Example [9]

Question

Construct ΔXYZ with XY=9cm, XYZ=45° and YXZ= 60°

Solution

Step 1- Construct a line XY=9cm and label the line XY. At point Y, construct an angle of 90° and bisect the angle to make an angle of 45°

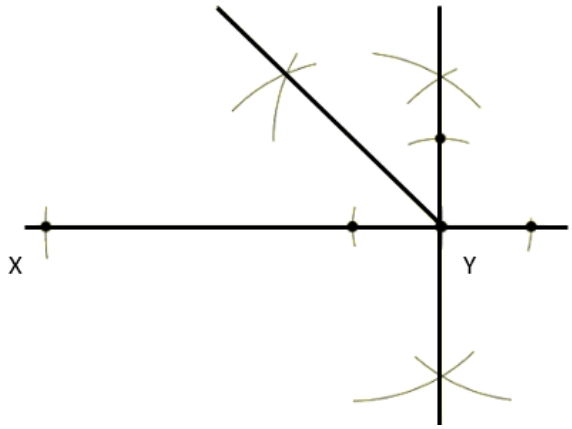


Fig 11.51

Step 2 - At point X, construct an angle of 60°. At the point of intersection, label Z

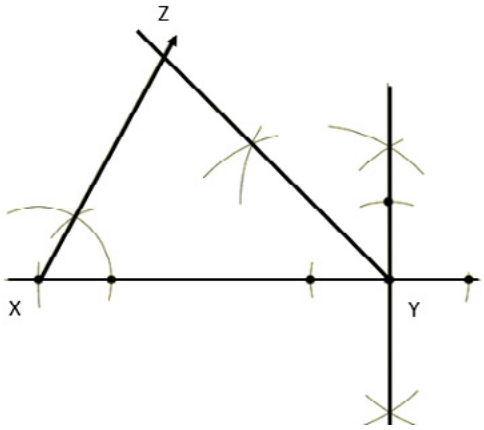


Fig 11.52

11.7 Summary

The unit looked at how to construct straight lines, bisecting straight lines, constructing angles, bisecting angles.

11.8 Further Reading

1. Macrae, M, Madungwe, L and Mutangadura, A (2017) New General Mathematics Book 3, Pearson Capetown.
2. Macral, M, Madungwe, L and Mutangadura, A (2017) New General Mathematics Book 4, Pearson Capetown.
3. Meyers, C; Graham, B; Dawe, L (2004) Mathscape Working Mathematically, 9th Ed, MacMillan
4. Pimentel R and Wall T (2011) International Mathematics, Hodder UK
5. Rayner D (2005) Extended Mathematics. Oxford New York

11.9 Assessment Test

1) Use ruler and compass only for all constructions and clearly show all construction lines and arcs on a single diagram

(a) Construct a quadrilateral ABCD in which AB = 4cm, BC=6cm, CD=5cm, ABC=135°and BCD=120° (6)

(b) Measure and write down

(i). The length of AD

(ii). BAD (2)

(c) Construct the locus of points

i. equidistant from AB and BC

ii. 3cm from BC and on the side of BC as A

iii. 4cm from B (5)

N2006/P2/#6

2) Use ruler and compass only. All construction lines and arc must be clearly shown. Construct on a single diagram,

i. A triangle XYZ in which XY=9,5cm, YXZ=45°and XZ=6,3cm

ii. the locus of points equidistant from X and Y,

iii. the locus of points equidistant from Y and Z,

iv. The circle passing through X,Y and Z
(10)

3) Use ruler and compasses only. All construction lines and arcs should be clearly shown. Three schools, P, Q and R are such that the bearing of Q from P is 045° and that of R from P is 300°. The distance between P and R IS 18km and Q is due east of R.

(a) (i). Using a scale of 1cm to represent 2km, construct a single diagram to show the relative positions of the 3 schools, P, Q and R.
(5)

(ii) Use the diagram to find the actual distance from P and Q.

(iii) Construct the perpendicular from R to QP produced

a. Calculate the area of the triangular region PQR, giving your answer in km^2

J2008/P2/#8

UNIT 12 - FINANCIAL MATHS 2

CONTENTS

12.1 INTRODUCTION

Have you ever been involved in monetary issues in your day to day activities? If it is a yes, how has it affected your daily life style? In this unit we are going to see how foreign exchange rates and taxes which include PAYE, VAT, customs and excise duty are calculated. We are also going to see how these affect our daily lives.

OBJECTIVES

After going through this unit, you should be able to:

- convert currencies from one form to another
- calculate sales and income tax
- solve problems related to PAYE, VAT, excise and customs duty
- read and intepret data represented in tables.

KEY TERMS

Net income: – is also called take home. It is the pay of an individual after deductions.

Gross income: – is one's salary before deductions.

P.A.Y.E: –is the tax which is deducted from salaries.

V.A.T: –is the tax levied on sales of goods and services.

TIME: You are advised not to spend more than 10 hours in this unit'

STUDY SKILLS

The key skill to mastery of mathematical concepts is practice. You need to solve as many Financial Mathematics related problems as possible for you to grasp all the concepts in this topic.

12.2 FOREIGN EXCHANGE

Our country has seen an increase in the number of cross boarder trading hence the need for one to be able to change money from one currency to another in order to trade effectively. Changing from one currency to another is called foreign currency exchange. These days currencies are rated against American Dollar (US$). Daily financial institutions such as banks, announce rates of currencies against the US Dollar.

The following examples provide us with the details on how currencies are converted.

Worked Example [1]

Questions

Given that R100 is equivalent to US$8.

a) Find the ratio of US$:Rands in the form 1:n
b) Peter wanted to exchange R250 into US$, how many US$ will he get from that amount?

Solutions

a) First write the ratio

US$8 : R100

US$1 : **less**

$= \frac{1}{8} \times 100$

= 12,5

$\therefore$ ratio is US$1: R12,5.

b) Using the ratio of

US$1 = R12,5

more = R250

$\frac{250}{12.5} \times$ US$1

= US$20

$\therefore$ Peter will get US$20 from R250.

Worked Example [2]

Questions

The table below shows rate of exchange of different currencies

Table1 2.1 – Exchange rates

CURRENCIES	Rate of exchange
Pula/ US$	1,25
Rand/Pula	0,7
US$/£	1,45

Key Pula/ US$ means US$1,25 = 1 Pula

Using the exchange rate above convert

a) US$5 to £
b) 7 Pula to Rands
c) R15 to Pula
d) 4 Pula to US$
e) 4 Pula to £

Solutions

a) From the table it means we are exchanging £1,45 with US $1.

£1,45 = US$1

more £ = US$5

$\frac{5}{1} \times$ £1,45

$=$ £7,25.

$\therefore$ US$5 $=$ £7,25.

b) From the table it means we are exchanging P0,7 with R1

P0,7 = R1

P7 = more rands

$\frac{7}{0,7} \times 1$=R10

$\therefore$ 7 Pula $=$ 10Rands.

c) Converting R15 to Pula

R1 = P0,7

R15 = **more**

$= \frac{15}{1} \times 0,7$

= P10,5

d) From the table, 1 Pula is exchanged with US$1,25

P1 = US$1,25

P4 = **more**

$= \frac{4}{1} \times 1,25,$

= US$5

e) Since there is no exchange rate of Pula to Pounds, we have to exchange Pula to US$, then from US$ to Pounds.

P4 = US$5, then

US$1 = £1,45

US$5 = **more**

$= \frac{5}{1} \times 1{,}45,$

= £7,25.

Activity (*12.1*) Foreign exchange

Questions

Using the exchange rate on **Table 12.1** answer the following questions

1) Convert R60 to:

a) Pulas b) US$ c) Pounds

2) Convert £140 to:

a) US$ b) Pulas c) Rands

3) Convert P200 to:

a) US$ b) Rands c) Pounds

Answers

1a) P42	b) US$52.50	c) £76.13
2a) US$96,55	b) P77,24	c) R110,34
3a) US$250	b) R285,71	c) £362,5

12.3 CONVERSION OF RATES AND CALCULATIONS

When you visit any bank, you will find a big billboard showing the exchange rates of different curriences mainly against the US$. The exchange rate vary from day to day.

The table below shows a billboard for a certain bank in US$

Table 12.2

CURRENCY	BUYING	SELLING
South Africa(ZAR)	1,563	1,670
Botswana(Pula)	1,252	1,301
Zimbabwe(RTGS$)	7,564	7,992
Zambia(Kwacha)	5,766	5,968

From the above table it means that:

- if you go to the bank with RTGS$ the bank will buy RTGS $7,564 with US$1 but,
- if you go to the bank with US$ the bank will sell RTGS$7,992 FOR US$1

Worked Example [3]

Questions

a) Rudo went to the bank to buy Rands with US$20. How many Rands did she get?

b) The bank then sells that same amount to Mrs Mashoko who wanted to buy a flight ticket in US$. How much pofit did the bank make?

Solutions

a) Since the exchange rate of buying rands in US$ is

R1,563 : US$1

US$20 = **more**

$=\frac{20}{1} \times 1{,}563$

$= \text{R}31{,}26$

b) The selling price is now

US$1:R1,67

so, $= 1{,}67 \times 20$

= R33,40

∴The bank made a profit of (R33,40 – R31,26)

= R2,14

Worked Example [4]

Question

Mrs Muchero wanted to export African attire fabrics from Zambia worth 6270 Kwacha. How much in US$ does she get from her bank if the bank charges 2% commision.

Solution

K627000 = $\frac{6270}{5{,}766} = \text{US\$ } 1087{,}41$

Less 2% commission= $\text{US\$ } 1087{,}41 - (\frac{2}{100} \times 1087{,}41)$

=US$1087,41−US$21,75

=US$1065,66

Worked Example [5]

Question

Kupakwashe wanted to exchange US$43,70 to Pulas. How much in Pula will he receive if the bank charges 1% commission?

Solution

US$43,70 $\times$ 1,301

= P$56,85, then

Less ($\frac{1}{100} \times 56{,}85$)

= P56,85−P0,569

=P56,28

Activity (*12.2*) Conversion of rates

Questions

1) Using information on **Table 12.2** using the bank's buying rate convert
   a) P1440 to US$
   b) RTGS $800 to US$
   c) 3786 Kwacha to US$
   d) R120 to US$
2) Using information on **Table 12.2** using the bank's selling rate convert
   a) P1440 to US$
   b) RTGS$800 to US$
   c) 3786Kwacha to US$
   d) R120 to US$
3) Bertha wanted to export fruits to South Africa worth R4500 with US$. The bank then charges 1,5% commision. How much money did she receive in US$?
4) Tatenda deposited 6789 Kwacha in his bank account but wanted the bank to rate it in US$. How much money does he receive in his bank account if the bank charges 2% commission?

Answers

1a) US$1150.16	b) US$105.76	c) US$656.61	d) US$76.78
2a) US$1106.84	b) US$100.10	c) US$634.38	d) US$71.86

3) US$2835.89

4) US$1153.87

12.4 SALES AND INCOME TAX RATES

12.4.1 P.A.Y.E (Pay As You Earn)

PAYE is money deducted from salaries or wages as tax on a daily, weekly, fortnightly, monthly or yearly basis.

The taxable amount is calculated after subtracting the exemptions and deductions from basic salary and allowances.

The table below shows monthly taxable PAYE

Table 12.3 - PAYE

TAX BRACKET			RATES
Less than 200	To	$200	5%
$201	To	$400	10%
$401	To	$600	12%
$601	To	$800	15%
$801	And	Above	20%

Worked Example [6]

Questions

Mr Allan's monthly salary is $ 645.

a) Calculate his PAYE.

b) Given that he has no tax free allowance or any deductions, calculate his HIV/AIDS levy if the levy is charged at 5% of his PAYE.

c) Calculate his net income.

Solutions

a) Mr Allan's salary is at the range ($601to $800) and charged 15% PAYE

$$\frac{15}{100} \times 645$$
$$= \$96{,}75$$

His PAYE is $96,75

b) HIV/AIDS levy =5% of $96,75

$$= \frac{5}{100} \times 96{,}75$$
$$=\$4{,}84$$

c) Net income =$645−($4,84+$96,75)

=$543,71

Worked Example [7]

Questions

Ms Mukuku is paid a monthly salary of $740 and she has the following deductions before tax:

Pension	$45,60
Medical Aid	$20,40
Doves funeral	$36,00
nssa contribution	$10,00

Ms Mukuku also receive the following tax free allowances

Housing	$70,60
Rural	$40,36

She is then levied 5% HIV/AIDS of her PAYE

Calculate her:

a) PAYE
b) AIDS levy
c) Net income salary

Solutions

a) Deductions= (45,60 + 20,40 + 36,00 + 10,00)

=$112

Taxable income=($740−$112)

= $628

$\therefore \text{PAYE} = \frac{15}{100} \times \$628$, ($628 falls under 15% in the monthly table)

$= \$94{,}20.$

b) AIDS levy

$= \frac{5}{100} \times \$94{,}20$

$= \$4{,}71.$

c) $740 + $70,60 + $40,36 − ($4,71 + $94,20 + $112)

=$850,96 − $210,91

=$640,05

Worked Example [8]

Questions

Mr George has a similar payslip similar to Ms Mukuku's but his monthly salary is $890. They had same deductions and same tax free allowances. Calculate his

a) P.A.Y.E
b) AIDS levy
c) his take home salary

Solutions

a) Taxable income

= $890 − $112

= $778

P.A.Y.E

$= \frac{15}{100} \times \$778$ (since $778 falls under 15% chargable rate)

$= \$116{,}70.$

b) AIDS Levy

$= \frac{5}{100} \times \$116{,}70$

$= \$5{,}84$.

c) Take home is the same as net income

$= \$890 + \$70{,}60 + \$40{,}36 - (\$5{,}84 + \$116{,}70 + \$112)$

$= \$1000{,}96 - \$234{,}54$

$= \$766{,}42$

12.5 VAT

VAT (Value Added Tax) is levied on sales of goods and or services and is usually calculated from the producer or manufacturer. There are some goods and services which might not be levied VAT and these include foodstuffs and non-profit making services.

Worked Example [9]

Question

Rudo bough a cosmetics set in the cosmetic shop for $33,40. The price displayed included 12% VAT. Calculate the price of the cosmetic set before VAT.

Solution

Price before VAT = 100%

Price after VAT = 112% = $33,40

$\frac{100}{112} \times \$33{,}40,$

$=\$29{,}82.$

Worked Example [10]

Questions

Buhle buys 10 suits from the supplier at $75 each for resale. He puts the markup of 15% on the suits. VAT is charged at 20%.

a) Calculate the selling price of the suits including VAT.

b) Calculate the profit made by buhle for each suit.

Solutions

a) Selling price before VAT

$= \$750 + (\frac{15}{100} \times 750)$

$= \$750 + \$112{,}50$

$= \$862{,}50.$

VAT at 20%

$= \frac{20}{100} \times 862{,}50$

$= \$172{,}50.$

Selling price including VAT

$= \$862{,}50 + \$172{,}50$

$= \$1035$

∴ It means each suit will be sold at \$103,50

b) Profit = Selling price − Buying price

$= \$103{,}50 - (\$75{,}00 \times \frac{120}{100})$

$= \$103{,}50 - \$90$

$= \$13{,}50$

Worked Example [11]

Questions

The table below shows an incomplete water bill for a certain month. Complete the blank spaces on the bill.

Table 12.4

Fixed charge	\$20,50
031647 to 031999 at 20c per unit	a) .....
Subtotal	b) .....
VAT at 20%	c) .....
Total amount due	d) .....

Solutions

a) From 031999 to 03220 = 352 units

b) 352 × \$0,2 = \$70,40

Subtotal = \$70,40 + \$20,50

= \$90,90

c) VAT $= \frac{20}{100} \times \$90{,}90$

$= \$18{,}18.$

d) Amount due \$90,90 + \$18,18 = \$109,08.

12.6 CUSTOMS AND EXCISE DUTY

This is money charged on goods or property and collected from visitors when entering the border of another country. The money goes to the government. Returning residence are also charged the duty on goods or items they buy, that is when they exceed allowed free duty amount. Zimbabwe used to allow its citizens to buy and bring goods worth US\$300 free duty per month.

Let's say one brings goods worth US\$400, at the border he or she will be charged duty (US\$400 – US\$300 free duty) of US\$100 at a rate stated by the government.

Worked Example [12]

Question

Ahmed bought goods worth \$500 outside the country. When he arrived at the border he was charged 20% duty. How much did he pay for duty?

Solution

Since Ahmed bought goods worth more than \$300, the duty will be charged for \$200 since \$300 is duty free.

Duty $= \frac{20}{100} \times \$200$

$= \$40.$

12.7 Summary

The unit looked at exchange rate and calculations involved changing from RTGS $ to any other currencies. We have also noted that exchange rates vary from day to day and normally all curriency exchange use the American dollar (US$). The unit also looked at caculations on VAT. VAT is levied on sales of goods and or services and is usually calculated from the producer or manufacturer. We have also looked at calculations involved on PAYE that is deducted from salaries or wages on a daily, weekly, fortnightly, monthly or yearly basis.

12.8 Further Reading

1. Macrae, M, Madungwe, L and Mutangadura, A (2017) New General Mathematics Book 3, Pearson Capetown.
2. Macral, M, Madungwe, L and Mutangadura, A (2017) New General Mathematics Book 4, Pearson Capetown.
3. Meyers, C; Graham, B; Dawe, L (2004) Mathscape Working Mathematically, 9th Ed, MacMillan
4. Pimentel R and Wall T (2011) International Mathematics, Hodder UK
5. Rayner D (2005) Extended Mathematics. Oxford New York

12.9 Assessment Test

Using the tax bracket on **Table 12.3** answer the questions 1 and 2.

1) Mrs Makoni receives a monthly salary of $270. She had the following deductions and allowances

Deductions		Allowances	
Pension	$3.10	Housing	$25.00
Medical Aid	$11.70	Transport	$36.00

She pays 5% HIV/AIDS levy every month.

Calculate

a) PAYE

b) HIV/AIDS Levy

c) Her net salary

2) Mr Chimuti works at a private company in the rural areas earning $1500. He has the following deductions and tax free allowances

ALLOWANCES		DEDUCTIONS	
Housing	$150,00	Pension	$45,00
Rural	$130,00	NSA contribution	$37,00
		Medical aid	$50,00
		Doves funeral	$75,00

Calculate

a) PAYE

b) His HIV/AIDS levy which is given as 5% of his PAYE

c) His take home salary

3) Nothando bought goods from the manufacturer worth $984,30 for resale excluding VAT. She charged 25% mark-up on her goods. VAT was charged at 10%.

a) Calculate the selling price of the goods including VAT.

b) Calculate the profit made by Nothando

4) Use **Table 12.4** to answer this question

The meter increased from 031999 to 032200 for the following month. The fixed charge and the rate of VAT has not changed

Calculate

a) Number of units used

b) Subtotal

c) VAT

d) Amount due

Answers

1a) PAYE=$25'52 b) AIDS Levy=$1.28 c) net salary $289.40

2a) PAYE=$258.60 b) AIDS Levy=$12.93 c) take home=$1301.47

3a) $1353.41 b) $270.68

4a) 201 b) $60.70 c) $12.14

d) $72.84

UNIT 13 - MEASURES AND MENSURATION 1

CONTENTS

13.1 INTRODUCTION

Plane and solid shapes make up a bigger percentage of objects that we use in everyday life. Let's think of kitchen utensils such as cups and pots, papers, manila charts and even our houses are good examples of plane and solid shapes. In this unit we are going to find areas, perimeters and volumes of such shapes and many other combined shapes.

OBJECTIVES

After going through this unit, you should be able to

- calculate perimeter and area of combined shapes
- calculate volume of cones and pyramid
- calculate density of objects
- calculate area and volume of similar figures
- calculate surface area and volumes of prisms and frustums

KEY TERMS

Area–the size of the surface of a shape, and that **area** is measured in square units

Perimeter – the distance around a shape.

Volume –is the number of cubic units that make up a solid figure.

Frustum – is a portion of a solid that lies between one or two parallel planes cutting it.

Prism–a solid shape that has the same cross-section all along the **shape** from end to end.

Altitude – this refers to perpendicular height between two points or between two parallel lines

TIME: You are expected not to spend more than 8 hours on this unit.

STUDY SKILLS

The key skill to mastery of mathematical concepts is practice. You need to solve as many problems in Measures and Mensuration as possible for you to grasp all the concepts in this topic.

13.2 PERIMETER OF COMBINED SHAPES

The perimeter of a shape is defined as the distance around the outside of a shape. The concept of perimeter depends much on the use of basic formulae of perimeter introduced at Level 1. However, combined shapes may require use of different types of formulae on one question.

Worked Example [1]

Questions

In the diagram below is a sheet of paper cut to give the dimensions in centimetres. You are required to find

a) the value of y

b) the Perimeter of the shape formed

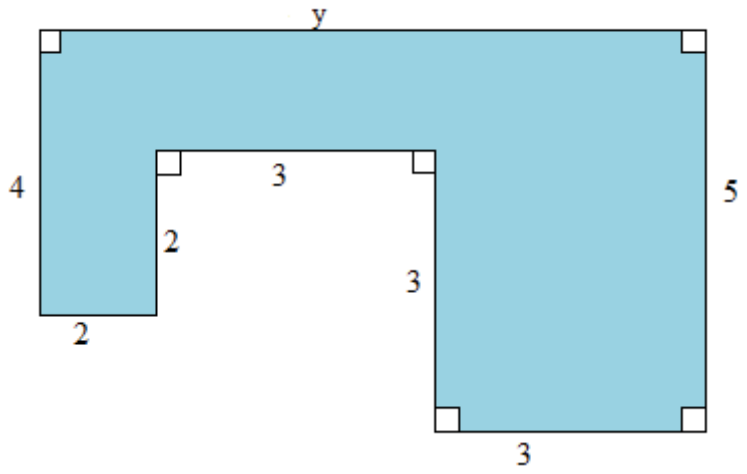


Figure 13.1

Solution

TIP: take note of the given sides which are parallel to side y

a) $y = 2 + 3 + 3 = 8\text{ cm}$

TIP: add all dimensions

b) $\text{Perimeter} = 2 + 2 + 2 + 3 + 3 + 5 + 8 + 4 = 29\text{ cm}$

Worked Example [2]

Question

In the diagram below are concentric semi-circles, the lengths are given in centimetres. Find the perimeter of the shape. Take π to be 3.142.

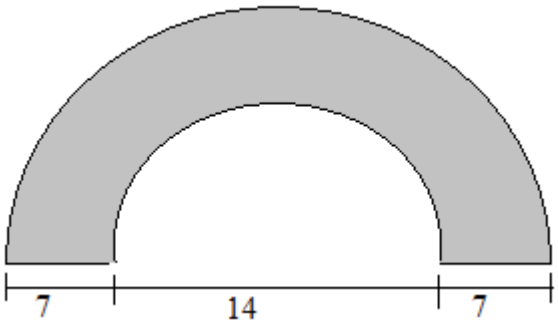


Figure 13.2

Solution

TIP: You may need to find the inner and outer arcs separately then add up your dimensions to find the perimeter.

<u>Remember ($C = 2\pi r$ or πd)</u>

Inner arc $= \frac{1}{2} \times \pi d$

$= \frac{1}{2} \times 3.142 \times 14$

$= 21.994$

Outer arc $= \frac{1}{2} \times \pi d$

$= \frac{1}{2} \times 3.142 \times 28$

$= 43.988$

Perimeter $= 21.994 + 43.988 + 7 + 7 = 79.982$ cm(make sure all dimensions are included in the final answer)

Activity (*13.1*) Perimeter of combined shapes

Questions

Calculate the perimeter of each shape below. All dimensions are in mm. Use the value 3.142 for π. O represents the centre of a circle. Broken lines should not be included in the calculations of perimeter.

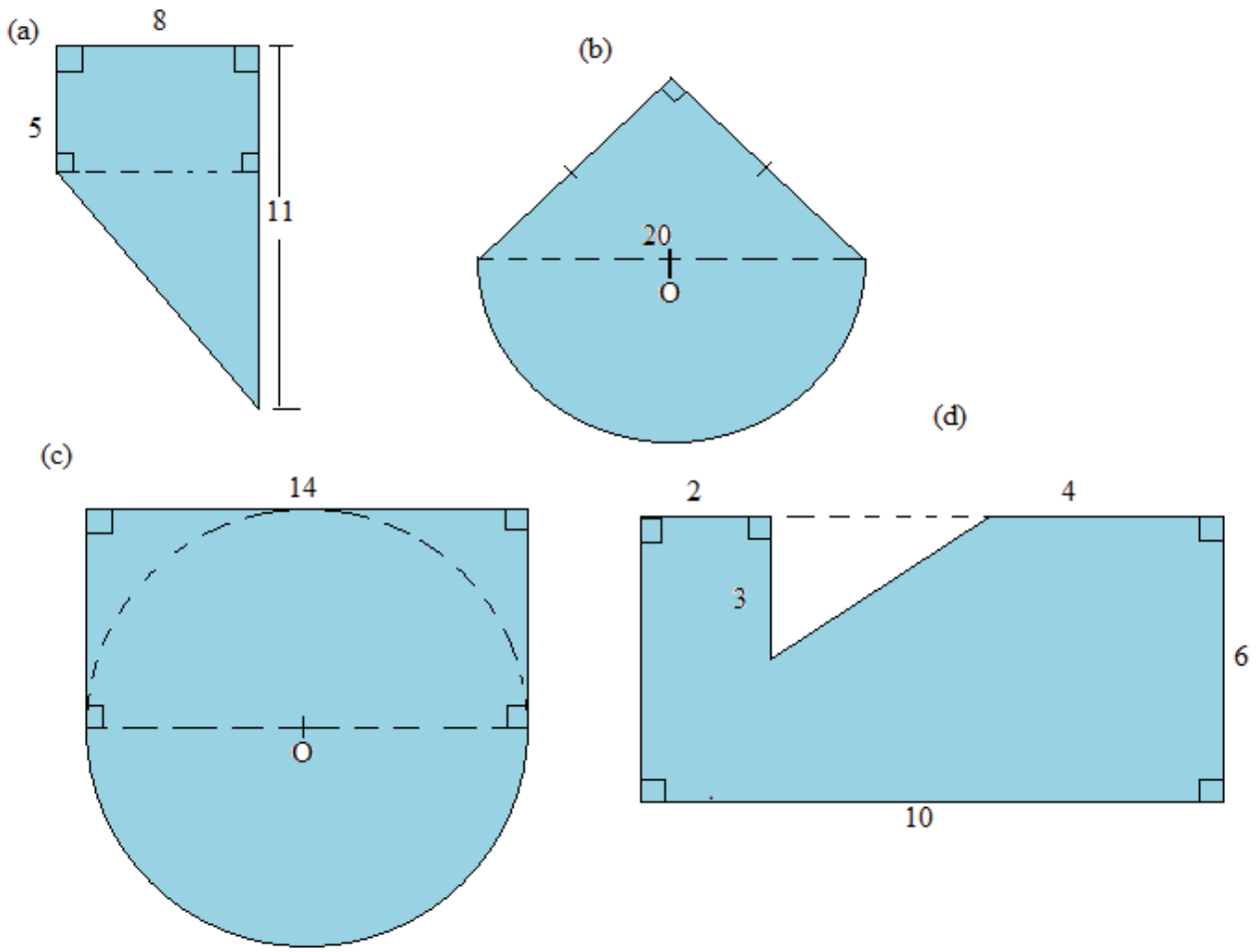


Figure 13.3

Answers

a) 34 b) 59.7 c) 49.994 d) 36

13.3 AREA OF COMBINED SHAPES

The area of a shape is defined as the amount of surface covered by the shape. You are required to recall formulae of plane shapes learnt in Level 1 and apply them where shapes are combined. Sectors will be dealt with separately so that you grasp the concept. Various forms of shapes will be used in examples.

Worked Example [3]

Question

In the diagram below is a combination of two equal triangles (A and C) and one rectangle (B). Find the area of the combined shapes.

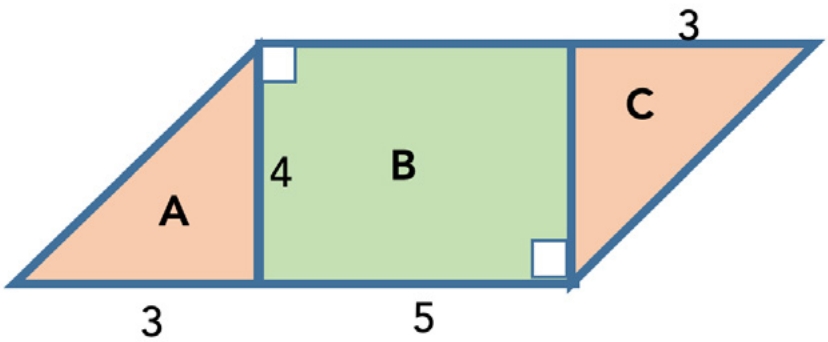


Figure 13.4

Solution

<u>Remember</u> (Area of $\Delta = \frac{1}{2}$bh and Area of rectangle = lb)

$$A + B + C = \left(\frac{1}{2} \times 3 \times 4\right) + (4 \times 5) + \left(\frac{1}{2} \times 3 \times 4\right)$$

$$= 6 + 20 + 6$$

$$= \underline{32 \text{ units}^2}$$

Worked Example [4]

Questions

The diagram below shows a combination of a semi-circle and a trapezium. Take π to be $\frac{22}{7}$. Find the area of the shape.

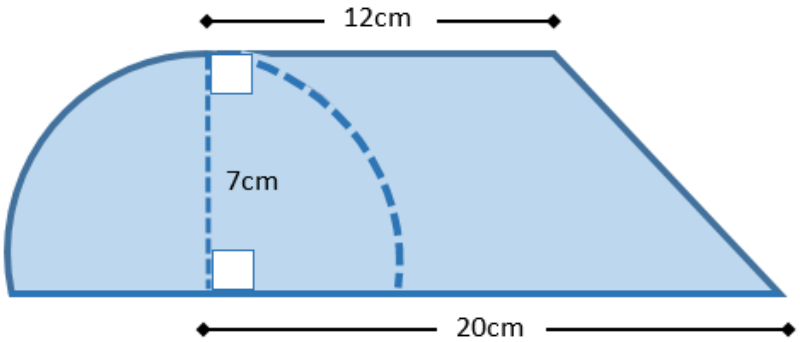


Figure 13.5

Solution

Remember (Area of trapezium $= \frac{1}{2}(a+b)h$ and Area of circle $= \pi r^2$)

Area = $\frac{1}{4}$ of Circle + Trapezium

$$= \frac{1}{4}\pi r^2 + \frac{1}{2}(a+b)h$$

$$= \frac{1}{4} \times \frac{22}{7} \times 7^2 + \frac{1}{2}(12+20) \times 7$$

$$= 38.5 + 112$$

$$\underline{= 150.5 \text{ cm}^2}$$

Let us include a much advanced example that involves other concepts in finding area of combined shapes.

TIP: You may need to understand the use trigonometry in finding dimensions

Worked Example [5]

Questions

The diagram below is a cross section of a dam wall made up of two shapes, that is triangle ABC and trapezium CXYZ. AB = 8m, AC = 3m. XY = 9m, YZ = 5m, CZ = 12m$B\hat{A}C$ = 60° and $C\hat{Z}Y$ = 30°. Find the cross sectional area of the dam wall.

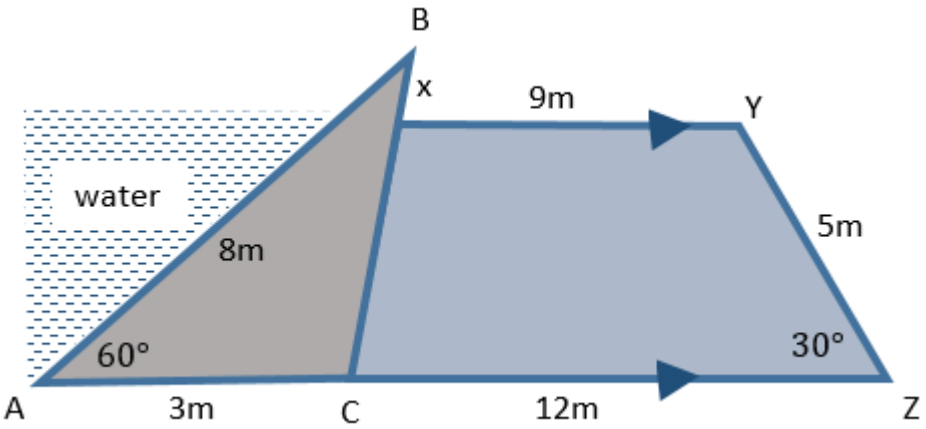


Figure 13.6

Solution

Let us start by finding the height of triangle ABC and the height of trapezium CXYZ.

Height of ΔABC is given by:$\text{Sin } 60° = \frac{h_1}{8}$

$$h_1 = 8 \times \sin 60°$$

Height of trapezium is given by: $\text{Sin } 30° = \frac{h_2}{5}$

$$h_2 = 5 \times \sin 30°$$

Cross-sectional area ABCXYZC = Area of Δ +Area of Trapezium

$$= \frac{1}{2}bh_1 + \frac{1}{2}(a+b)h_2$$

$$= \frac{1}{2} \times 3 \times 8 \times \sin 60° + \frac{1}{2}(9+12) \times 5 \times \sin 30$$

$$= 10.3923 + 26.25$$

$$= 36.6423$$

$$\underline{= 36.6 \text{ m}^2}$$

Now that we have gone through some examples, you should be ready to attempt the following activity.

Activity (*13.2*) Area of combined shapes

Questions

Calculate the area of each shape below. All dimensions are in cm. Use the value $\frac{22}{7}$for π. O represents the centre of a circle. You are required to recall the formulae for plane shapes.Where the diagram is shaded, you should find the area of the shaded sections only.

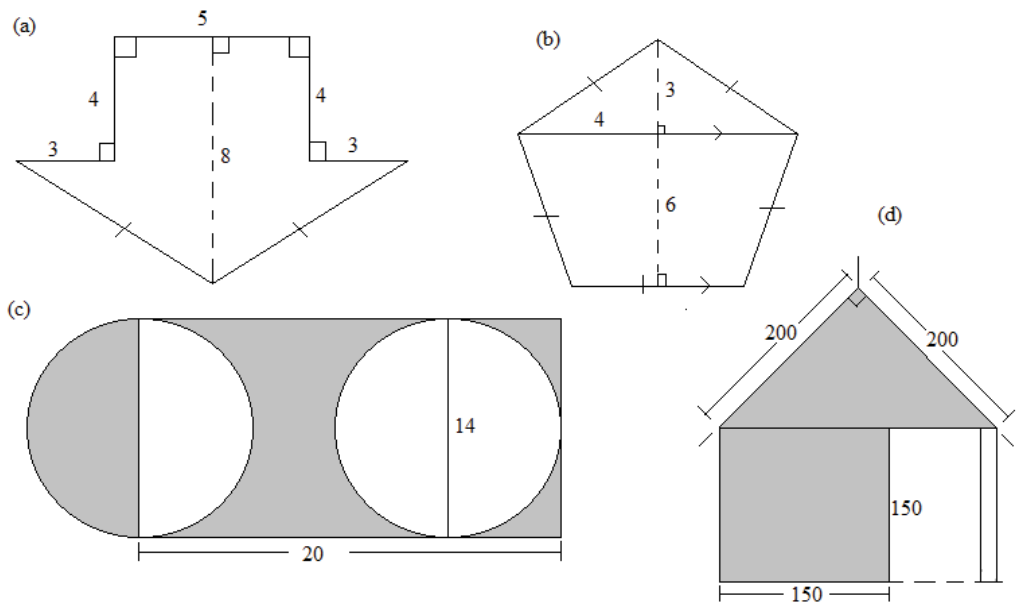


Figure 13.7

Answers

a) 42 b) 51 c) 126.04 d) 42 500

13.3.1 Parallelograms and triangles between parallel lines

The concept on combined areas of plane shapes is incomplete without inclusion of areas of parallelograms and triangles between parallel lines. Two theorems covering this concept will be stated and illustrated below.

❖ Theorem 1

Parallelograms on the same base between the same parallel lines are equal in area.

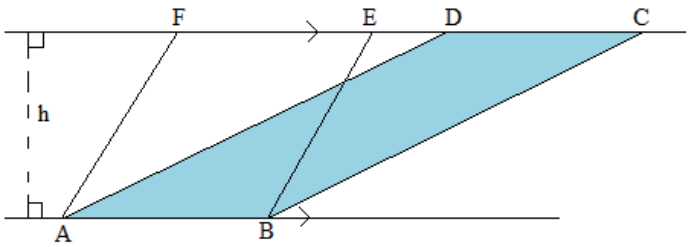


Figure 13.8

By the given theorem the area of a parallelogram ABEF = the area of parallelogram ABCD since they are sharing the same base AB and the same altitude h.

❖ Theorem 2

Triangles on the same base between the same parallel lines are equal in area

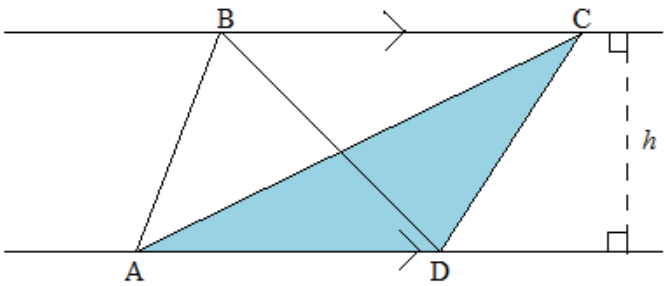


Figure 13.9

Using the given theorem, area of triangle ABD = area of triangle ACD since they are sharing the same base and the same altitude.

Worked Example [6]

Question

Identify parallelograms with equal areas from the diagram below

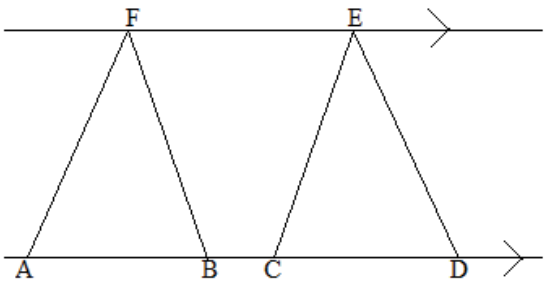


Figure 13.10

Solution

Using Theorem 1, parallelogram AFEC =area of parallelogram BFED,

Worked Example [7]

Questions

In the diagram below, identify the triangle equal in area to triangle ADC

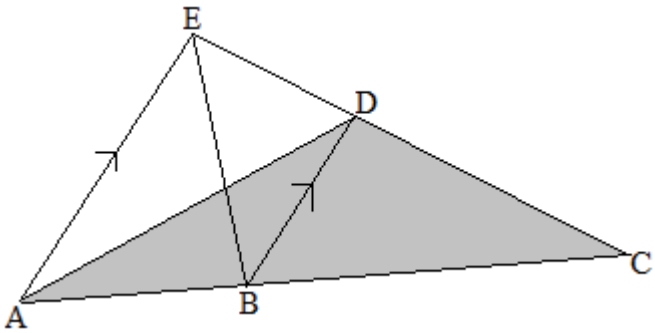


Figure 13.11

Solution

TIP: Area of triangle ADC = area of triangle AEC – area of triangle ADE

Now the same applies to:

Area of triangle BCE = triangle AEC – triangle ABE

Therefore, Area of triangle ADC = Area of triangle BCE(since$\Delta ADE = \Delta ABE$)

Activity (*13.3*) Parallelograms and triangles

Questions

Name a triangle equal in area to the shaded

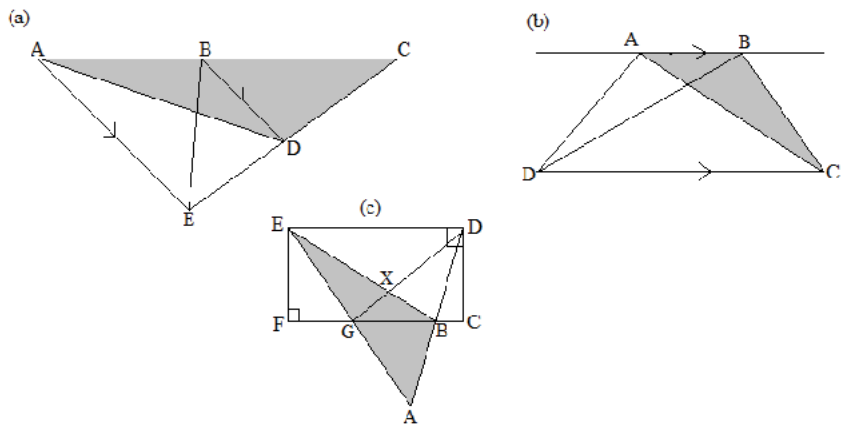


Figure 13.12

Construct any parallelograms equal in area, make sure you include parallel lines. As indicated in the answer space.

Answers

a) ΔBCE b) ΔBAD c) ΔAGD

13.3.2 Arcs, sectors and segments of circles

In this section we are going to apply the formulae for the lengths of arcs, areas of sectors and segments in circles.

Length of arc

The length of an arc of a circle is proportional to the angle at which the arc subtends at the centre.

In figure 13.5 below are examples of arcs formed from various angles on one circle.

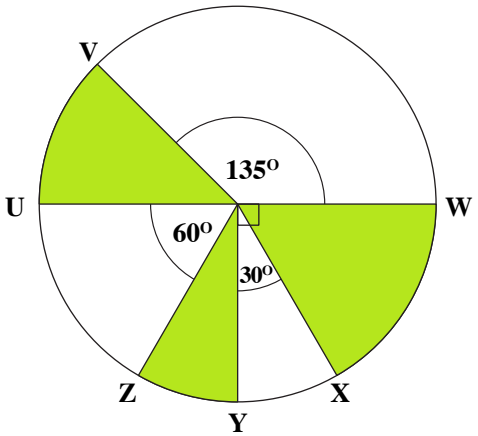


Figure 13.13

From the above diagram, it can be noted that the arc VW is $\frac{135}{360}$ or $\frac{3}{8}$ of the circumference of the circle.

Also the arc UZ is $\frac{60}{360}$ or $\frac{1}{6}$ of the circumference. Now, the formula for the length an arc can be developed as follows

Arc length $= \frac{\theta^o}{360^o} \times \text{circumference}$

Arc length $= \frac{\theta^o}{360^o} \times 2\pi r$

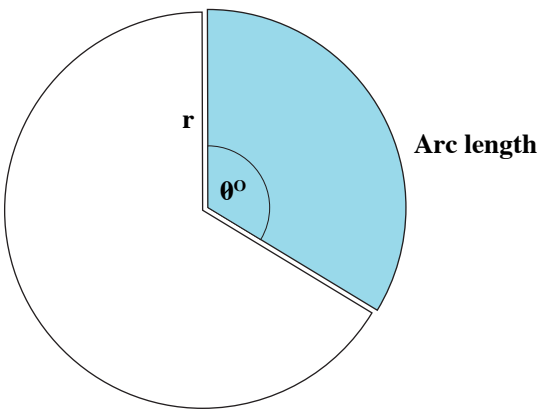


Figure 13.14

Let us look into some examples on how the arc length is calculated.

Worked Example [8]

Questions

An arc subtends an angle of 120° at the centre O of a circle. The radius of the circle is 12cm. take π to be $\frac{22}{7}$. Find the length of an arc of the circle.

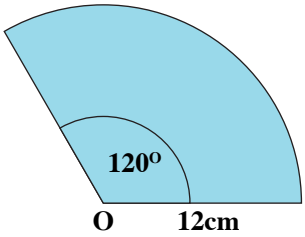


Figure 13.15

Solution

$$\text{Arc length} = \frac{120^o}{360^o} \times 2 \times \frac{22}{7} \times 12$$

$$= \frac{120^o}{360^o} \times 2 \times \frac{22}{7} \times 12$$

$$= 25.1(3 \text{ s.f})$$

Sometimes you can be asked to calculate the angle at which a certain arc length subtends at the centre of a circle. Let us look into such an example.

Worked Example [9]

Question

Calculate the angle at which an arc of length 44m subtends at the centre of a circle with a radius of 28m.

Solution

Sketch a diagrammatic representation of the information

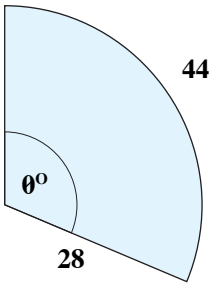


Figure 13.16

Use the arc length formula $= \frac{\theta^o}{360^o} \times 2\pi r$

$44 = \frac{\theta^o}{360^o} \times 2 \times \frac{22}{7} \times 28$

$\theta^o = \frac{44 \times 360 \times 7}{2 \times 22 \times 28}$

$= 90°$

Worked Example [10]

Question

An arc subtends an angle of 54° at the circumference of a circle with a radius of 3.5cm. Calculate the length of the arc (x). Take π to be 3.142

Solution

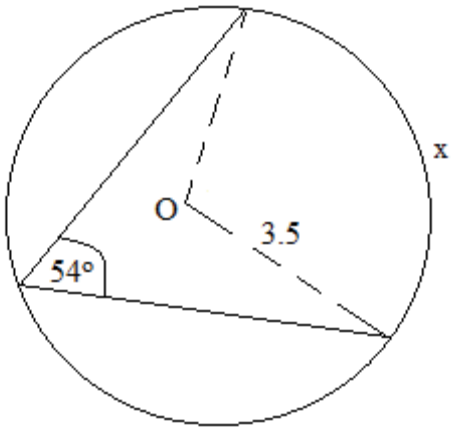


Figure 13.17

TIP: note that the angle at the centre, which subtends arc x is twice the angle 54°. The angle is 108°

Then, arc length $= \frac{108^o}{360^o} \times 2 \times \frac{22}{7} \times 3.5$

$= 6.6$ cm

Activity (*13.4*) Arc length

Questions

1. Calculate the length of an arc that subtends an angle of 35° at the centre of a circle with radius of 2.1cm. Give your answer in terms of π
2. Calculate the length of an arc that faces an angle of 150° at the centre of a circle with a radius of 6.3m. Take π to be $\frac{22}{7}$
3. A plastic ruler that is 30cm long is bent into an arc of a circle with a radius of 7cm. What angle does the ruler subtends at the centre of the circle. Take π to be $\frac{22}{7}$.
4. A singing choir stands in an arc of 28m and subtends an angle of 30° from the centre of circle where the choir master is standing. In the same circle, which angle does the arc of 35 m subtend?
5. In a circular fish pond of radius 9m, a division which makes are chord of 4m from the centre is drawn.
   (a) Calculate the angle subtended by the chord (division) at the centre of the circle.
   (b) Hence find the length of the major arc that is cut off by the chord
6. In the diagram below, the diameter of the circle is 28. Calculate the length of the major arc AB

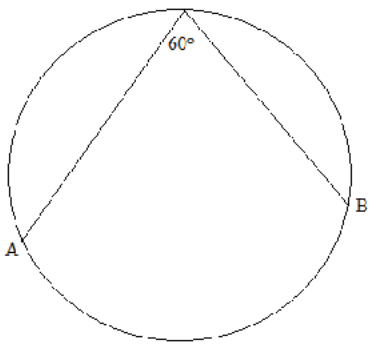


Figure 13.18

Answers

1) 0.408 π
2) 16.5
3) 245.5°
4) 37.5
5) a) 127.2° b) 36.6
6) $58\frac{2}{3}$

13.3.3 Perimeter of a sector

Figure 13.6 below shows a sector and its parts (properties). In this section we are going to develop and apply a formula for perimeter of a sector

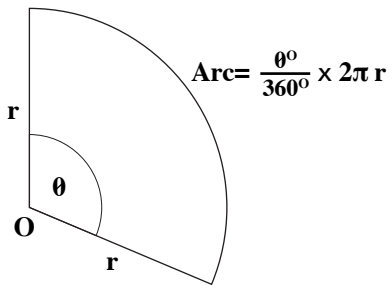


Figure 13.19

It must be clear that we are simply adding up the length of an arc and two radii to obtain the Perimeter

Perimeter of a sector $= \frac{\theta^o}{360^o} \times 2\pi r + 2r$

Worked Example [11]

Question

Calculate the perimeter of a sector whose angle θ at the centre is 120° and the radius is 0.7m. Take π to be $\frac{22}{7}$

Solution

A sketch of the diagram is given below

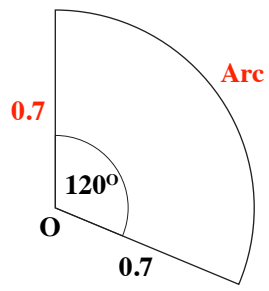


Figure 13.20

Perimeter of a sector $= \frac{120^o}{360^o} \times 2 \times \frac{22}{7} \times 0.7 + 2 \times 0.7$

$$= \frac{22}{15} + 1.4$$

$$= \frac{43}{15} = 2\frac{13}{15} \text{ m}$$

13.3.4 Area of Sector

The area of a sector of a circle is proportional to the angle of the sector. In figure 13.7 below the formula for area of a sector is developed, explained and applied

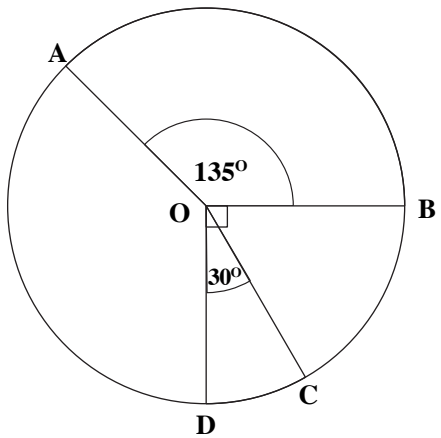


Figure 13.21

If we consider sector AOB we realise that the sector occupies $\frac{135}{360}$ or $\frac{3}{8}$ of the whole circle and if we consider sector COD we note that it occupies $\frac{30}{360}$ or $\frac{1}{12}$. Therefore,

Area of sector $= \frac{\theta}{360} \times \pi r^2$

Let us explore several examples below.

Worked Example [12]

Question

A sector whose angle at the centre is 150° and with a radius of 6.3cm is removed from a complete circle. Find the area of the sector as a multiple of π

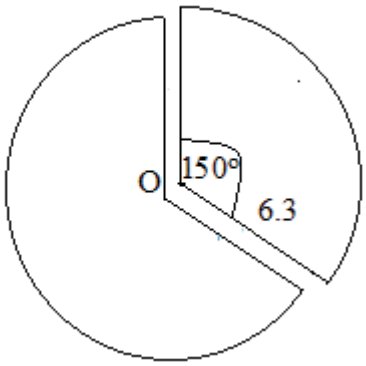


Figure 13.22

Solution

Area of sector $= \frac{150}{360} \times \pi \times 6.3^2$

$= \frac{5}{12} \times \pi \times 6.3 \times 6.3$

$= 16.5375\pi \text{cm}^2$

13.3.5 Area of segment

The area that lies between the minor arc and the chord of a sector is called a segment.

Area of segment = Area of sector – Area of triangle. As shown in the example below.

Worked Example [13]

Question

Find the area of the shaded segment in the diagram below. Take π to be $\frac{22}{7}$.

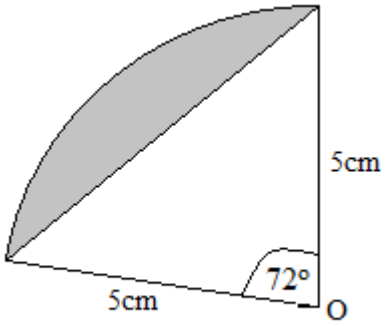


Figure 13.23

Solution

Area of segment $= \frac{72}{360} \times \frac{22}{7} \times 5^2 - \frac{1}{2} \times 5 \times 5 \sin 72°$

$= \frac{1}{5} \times \frac{22}{7} \times 25 \ - \frac{1}{2} \times 25 \times 0.9511$

$= 3.8261$

$= 3.83\text{cm}^2$

Activity (*13.5*) Perimeter, area and segments

Questions

Take π to be $\frac{22}{7}$ unless specified in the question

1. A sector has an angle of 108° and a radius of 2.1m.Find
   a) the perimeter of the sector
   b) the area of the sector
   c) the area of the segment
2. Calculate the area of the shaded parts in the diagrams below. O represents the centre of a circle

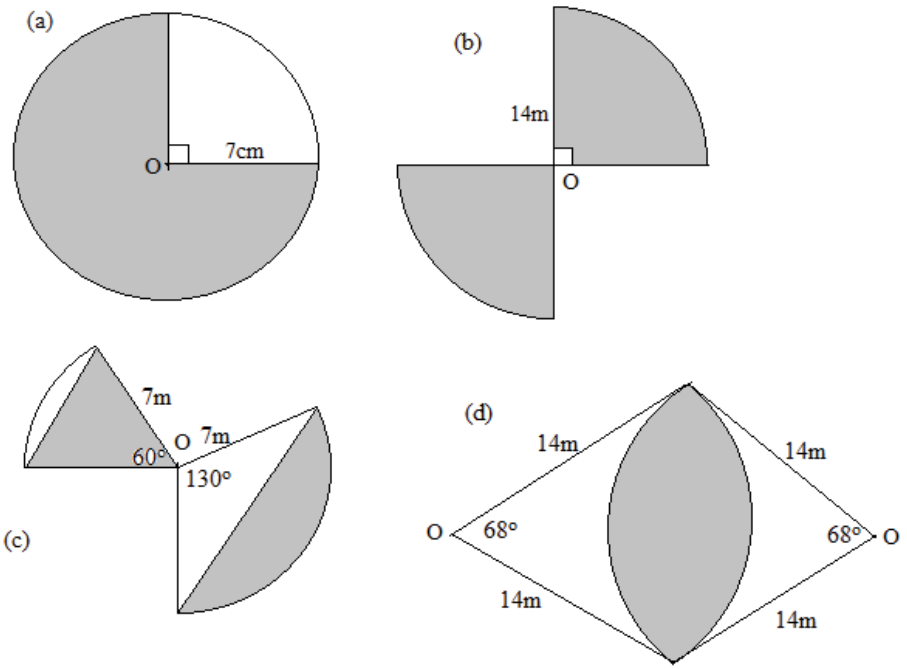


Figure 13.24

3. Calculate the perimeters of the shapes below. Broken lines are not part of the perimeter in question

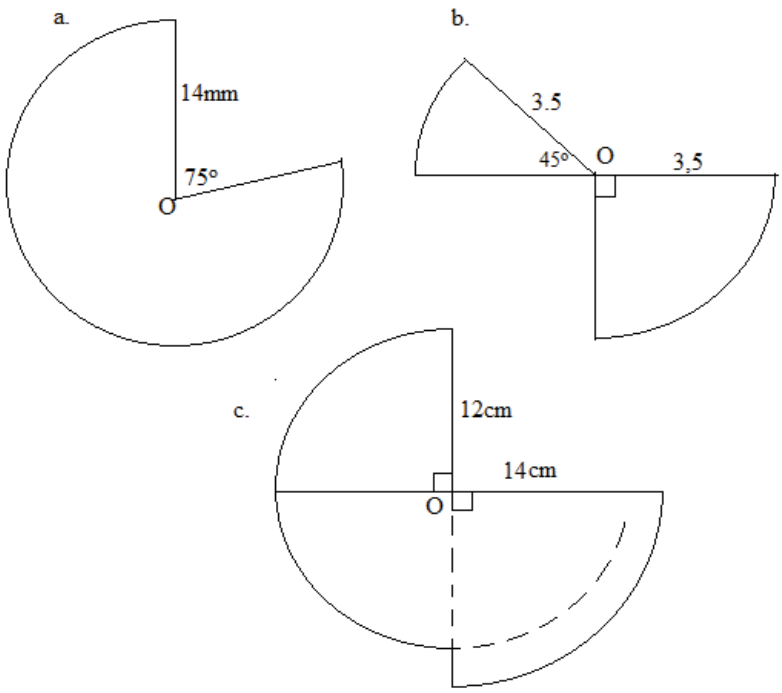


Figure 13.25

Answers

1) a) 3.96 b) 4.158 c) 2.06
2) a) 115.5 b) 308 c) 58.1

3) a) $97\frac{2}{3}$ b) 22.25 c) $87\frac{5}{7}$

13.4 VOLUME OF CONES, CYLINDERS AND PYRAMIDS

13.4.1 Volumes of cones and pyramids

These two types of solid shapes have a common general formula for volume, that is

Volume $= \frac{1}{3} \times$ base area $\times$ perpendicular height

Remember (Area of circle $= \pi r^2$)

Cones

Note that the base of a cone is circular.

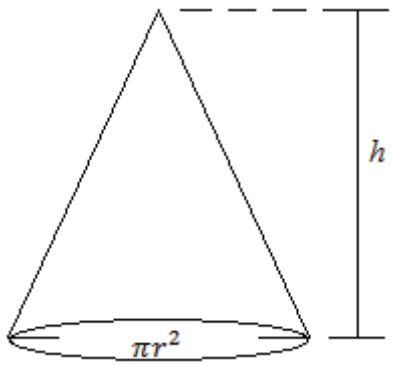


Figure 13.26

Therefore, Volume of a **cone** $= \frac{1}{3} \times \boldsymbol{\pi r^2} h$

Worked Example [14]

Question

A cone has a circular base with a radius of 7mm and a perpendicular height of 30mm. Find the volume of the cone as a multiple of $\boldsymbol{\pi}$

Solution

Volume of a **cone** = $\frac{1}{3} \times \boldsymbol{\pi} \times \mathbf{7^2} \times 30$

$= \boldsymbol{\pi} \times \mathbf{49} \times 10$

Volume of a **cone** = $490\boldsymbol{\pi}\ \mathbf{mm^3}$

Worked Example [15]

Question

In the diagram below is a cone with a slant height of 10m, base radius of 6m and a perpendicular height of h. Find the volume in terms of $\boldsymbol{\pi}$.

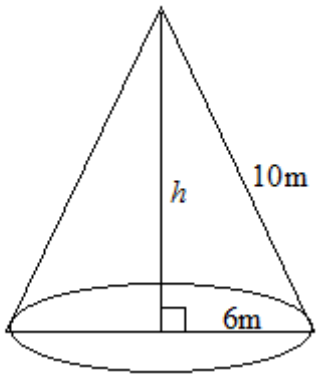


Figure 13.27

Solutions

$h^2 + 6^2 = 10^2$

$h^2 + 36 = 100$

$h = \sqrt{64}$

$h = 8\ m$

Volume of a **cone** = $\frac{1}{3} \times \boldsymbol{\pi} \times \mathbf{6^2} \times 8$

$= \frac{1}{3} \times 288\boldsymbol{\pi}$

$= 96\boldsymbol{\pi}\ \mathbf{m^2}$.

13.4.2 Volume of Pyramids

Pyramids vary in their base areas, you will note that some are rectangular based, triangular based or square based. But what is best is to be able to find the base area and apply the formulae accordingly.

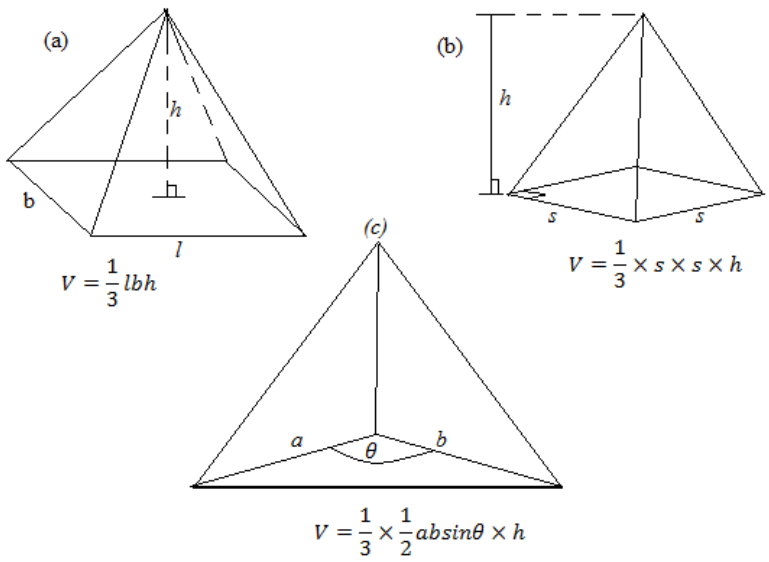


Figure 13.28

Worked Example [16]

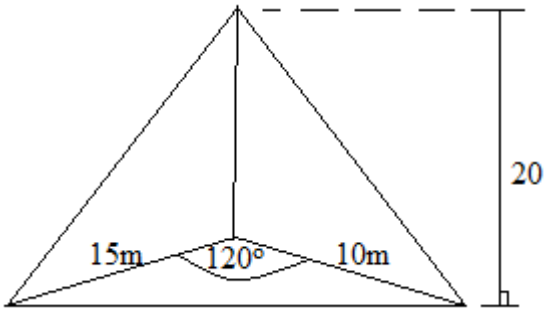
Questions

Give your answers to 3 s.f

(a) A pyramid has a square base of sides 4m each and a perpendicular height of 10m. Find its volume.

(b) A pyramid has a rectangular base of length of 7m, breath of 3m and a perpendicular height of 15m. Find its volume.

(c) A pyramid shown below has a triangular base. You are required to find its volume.



13.29

Solutions

(a) $V = \frac{1}{3} \times 4^2 \times 10 = 53\frac{1}{3}\,m^3$

(b) $V = \frac{1}{3} \times 7 \times 3 \times 15 = 105\ m^3$

(c) $V = \frac{1}{3} \times \frac{1}{2} \times 10 \times 15\text{Sin}120° = 21.6\ m^3$

13.4.3 Volume of cylinders

A cylinder is good example of a circular based prism where the volume is simply obtained by multiplying the base area and the perpendicular height of the cylinder.

Volume of a prime= base area × perpendicular height

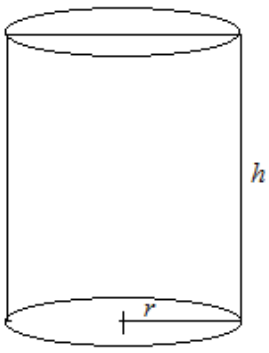


Figure 13.30

Volume of a **cylinder** = $\boldsymbol{\pi r^2}h$

Worked Example [17]

Question

A cylinder has a circular base with a radius of 42mm and a height of 1000mm. Take π to be $\frac{22}{7}$ and find the volume in cm^3

Solution

Volume of a **cylinder** = $\boldsymbol{\pi r^2}h$

$$= \frac{\mathbf{22}}{\mathbf{7}} \times \mathbf{42^2} \times 1000$$

$$= 5544000mm^3$$

$$= \frac{5544000}{10} cm^3$$

$$= 554400cm^3$$

Activity (*13.6*) Cones, pyramids and cylinders

Questions

Take π to be $\frac{22}{7}$ and give your answers correct to 3s.f where necessary

1. A conical tower of height 23m has a circular base of diameter of 105m. Find its volume.
2. A cone has a circular base of circumference 44mm and a perpendicular height of 50mm. Find its volume.
3. Calculate the volume of the pyramid in the diagram below

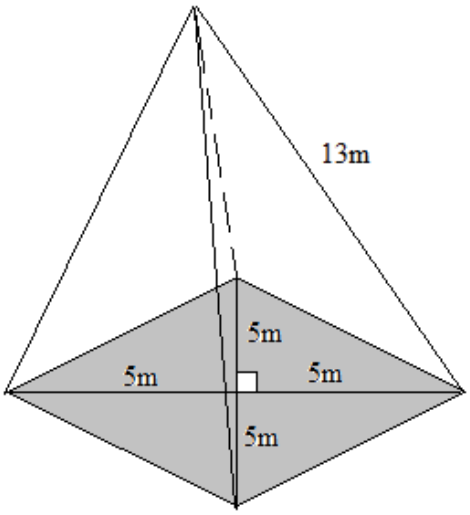


Figure 13.31

4. Calculate the volume of the cylinder in diagram below

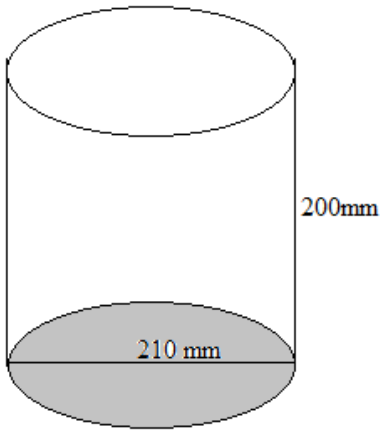


Figure 13.32

Answers

1) 66412.5 m^3
2) 2566.7 m^3
3) 200 m^3
4) 6930000 mm^3

13.5 DENSITY

Let us look at formulas related to density

Density $= \frac{\text{Mass}}{\text{volume}}$

Mass = Density ×Volume

Volume $= \frac{\text{Mass}}{\text{Density}}$

Worked Example [18]

Question

A metal has a density of 7g/cm^3. Find the mass in kgs of a metallic ball with a volume of 200cm^3.

Solution

Mass$= 7\text{g/cm}^3 \times 200\text{cm}^3$

$= 7 \times 200$

$= 1400\text{g}$

$= \frac{1400}{1000}$ kgs

$= 1.4\text{kgs}$

Activity (*13.7*) Density

Questions

Give your answers correct to 3 s.f.

1. A tin of beans has a base area of 154 cm^2, a height of 10cm and a mass of 100g. Find its density in g/cm^3.
2. A short-putt implement has a density of $0.8kg/cm^3$. If its mass is 900g. Find its volume in cm^3
3. An under 17girls javelin has a total mass of 700g and a volume of $800cm^3$. Find its density in kg/m^3
4. A dictionary has a density of $0.2g/cm^3$and has a mass of 2kgs. Find its volume in m^3

Answers

1) 0.649 g/cm^3
2) 1.125 cm^3
3) 0.000875 kg/m^3
4) 10000 m^3

REFLECTION

- Finding the area of a single or combined plane shapes depend upon specific formula, therefore you need to recall most the formulae.
- Solution for volume will always depend on the base area of a plane shape.
- Apply the change of subject method to find formulae for mass, density and volume

13.6 Summary

In this unit we have covered the aspects of perimeter and area of combined shapes. We have also made it clear on the solution of volume of Cones, Pyramid and cylinders. Note that we have factored in the concept on combined areas of triangles and parallelograms on parallel lines.

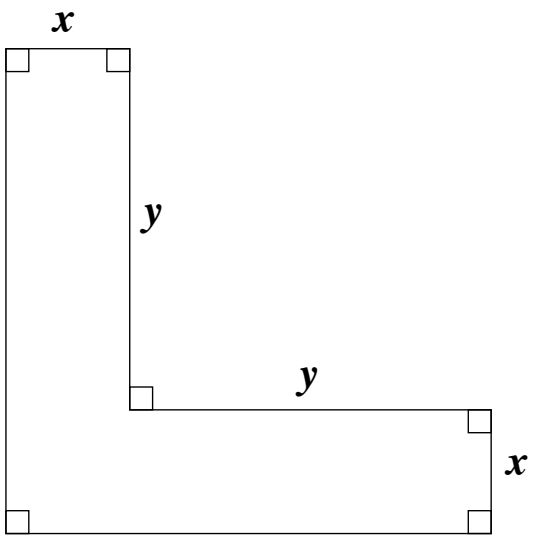
13.7 Further Reading

1. Macrae, M, Madungwe, L and Mutangadura, A (2017) New General Mathematics Book 3, Pearson Capetown.
2. Macral, M, Madungwe, L and Mutangadura, A (2017) New General Mathematics Book 4, Pearson Capetown.
3. Meyers, C; Graham, B; Dawe, L (2004) Mathscape Working Mathematically, 9th Ed, MacMillan
4. Pimentel R and Wall T (2011) International Mathematics, Hodder UK
5. Rayner D (2005) Extended Mathematics. Oxford New York

13.8 Assessment Test

Your assessment test will be based on actual ZIMSEC past exams, with exception of question 4. You are expected to write this test in not more than 45minutes

1) June 2004 QN2(a) paper 2

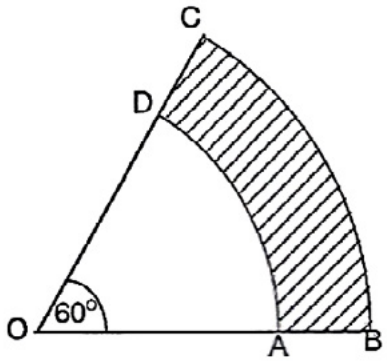


The diagram above represents an L-shaped flower bed. The dimensions are in metres. Write down and simplify, in terms of x and y, the expression for

(i) the perimeter of the flower bed,

(ii) the area of the flower bed. [3]

2) June 2004 QN7(a) paper 2



In this question take π to be $\frac{22}{7}$.

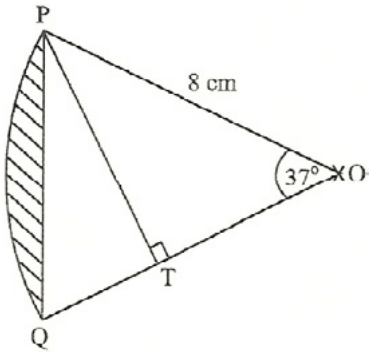
In the diagram, OAD and OBC are sectors of concentric circles centre O.
OA = 6 cm, OB = 7 cm and $C\hat{O}B = 60°$.

Calculate

(i) the area of sector OBC, [2]

(ii) the area of the shaded part. [3]

3) June 2011QN 3(b) paper 2



Take π to be 3,142.

A The diagram shows a sector of a circle centre O. $P\hat{O}Q = 37°$, PO= 8 cm and PT is perpendicular to QO. Calculate

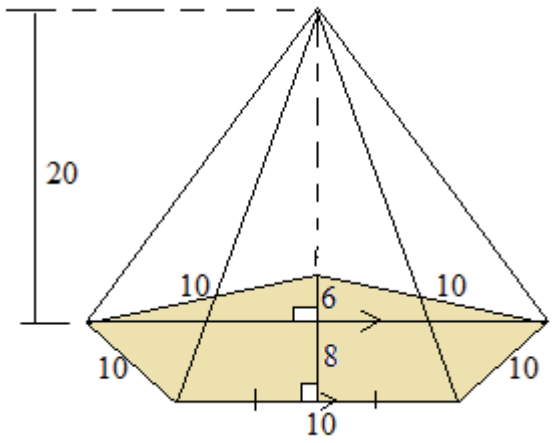
(i) PT,

(ii) area of the sector,

(iii) area of triangle PQO,

(iv) the area of the shaded segment. [7]

4) In the shape below is a cone with regular pentagonal base all dimensions are in cm. Calculate the volume.



UNIT 14 - MEASURES AND MENSURATION 2

CONTENTS

14.1 INTRODUCTION

Measures and Mensuration of plane shapes takes a further and more realistic approach in this unit where you are going to explore the plane shapes that make up solid shapes.

OBJECTIVES

After going through this unit, you should be able to

- calculate surface area of solid shapes
- calculate volumes of prisms
- calculate volumes of frustums

KEY WORDS

Surface area–the size of the surface of a shape, and that **area** is measured in square units

Exposed surface area –the size of a solid shape that can be seen from outside

Volume –is the number of cubic units that make up a solid figure.

Frustum – may be formed from a right circular **cone or a pyramid** by cutting off the tip of the **cone or pyramid** with a cut perpendicular to the height, forming a lower base and an upper base that are circular and parallel.

Prism–a solid shape that has the same cross-section all along the **shape** from end to end.

TIME: You are expected not to spend more than 8 hours on this unit.

STUDY SKILLS

The key skill to mastery of mathematical concepts is practice. You need to solve as many problems in Measures and Mensuration as possible for you to grasp all the concepts in this topic.

14.2 SURFACE AREA

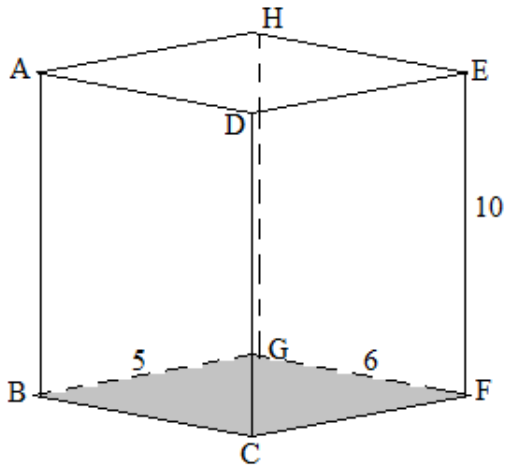
In this concept you are expected to recall almost all formulae for area of plane shapes learnt in Level 2 and Measures and Mensuration covered in Unit 13. We will be assisting you in our examples to recall these formulae.

Worked Example [1]

Question

In the diagram below is a cuboid ABCDEFGH with its dimensions given in mm.

Find the total surface area of the shape.



Solution

Remember (area of a rectangle = lb**)**

The total surface area is simply obtained from the sum of areas of rectangles which make up the surface of the shape.

Area of rectangle ABCD = Area of rectangle EFGH = 6 ×10 = 60

Area of rectangle ABGH = Area of rectangle CDEF = 5 × 10 =50

Area of rectangle BCFG = Area of rectangle ADEH = 5 × 6 = 30

Now each surface appears twice thus,

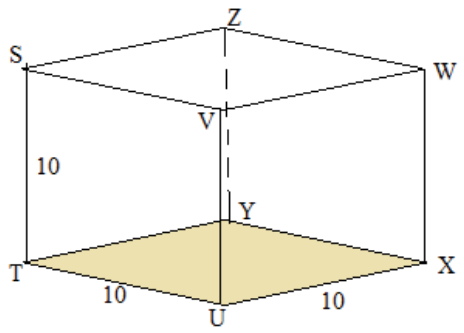
Total Surface area = $2(60 + 50 + 30)$ = 280 mm^2

Let us explore another simplified example

Worked Example [2]

Question

The diagram below shows a cube SUVWXYZ of side 10m each. Find the total surface are of the shape.



Solution

Remember (area of a Square = $s \times s$)

This one is much simpler because we are going to add 6 equal surfaces, that is

Total surface= $6 \times s \times s$

$= 6 \times 10 \times 10$

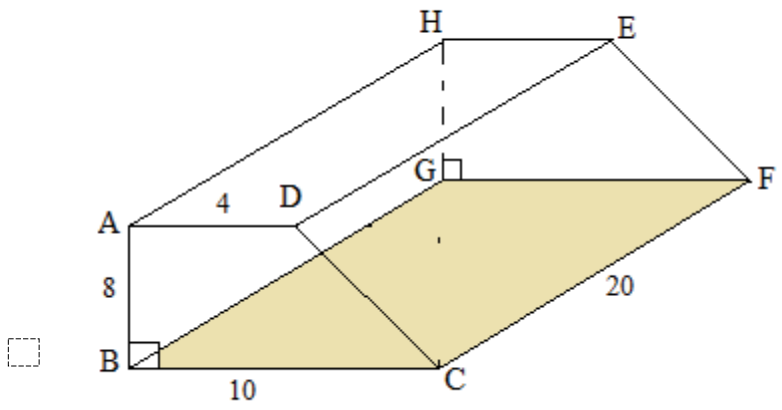
Total surface = 600m^2

Now let us consider another different example that will help to understand this concept in a better way

Worked Example [3]

Question

The diagram below is a trapezium based prism with all dimensions given in mm. You are required to find the total surface area of the shape.



Solution

TIPS:

- Know the plane shapes that make up the prism
- Know the formulae area of each type of plane shape identified

Area of trapezium ABCD $= \frac{1}{2}(a+b)h = \frac{1}{2}(4+10)8 = 56$

Area of trapezium EFGH $= \frac{1}{2}(a+b)h = \frac{1}{2}(4+10)8 = 56$

Area of rectangle ABGH $= 8 \times 20 = 160$

Area of rectangle BCFG$= 10 \times 20 = 200$

Area of rectangle ADEH $= 4 \times 20 = 80$

There is rectangle CDEF with a missing dimension CD. Apply the Pythagorean concept to obtain CD as follows.

$CD^2 = 8^2 + 6^2 = 100$

CD $= \sqrt{100} = 10$, now

Area of rectangle CDEF $= 10 \times 20 = 200$

Therefore the total surface area= $56 + 56 + 160 + 200 + 80 + 200 = 752\ \text{mm}^2$

14.2.1 Total surface area of a cone

It is vital that we explore the surface area of a cone by reducing the curved part to simple triangle and the base is just a circle. Figure 14.1 shows how the surface area is derived

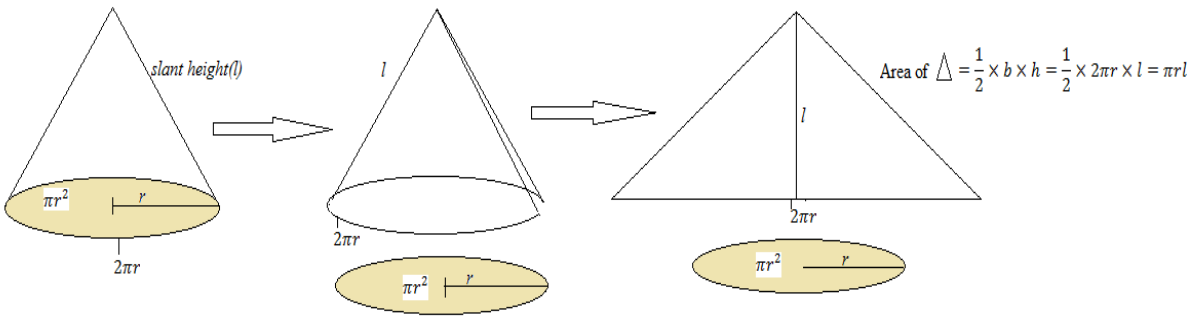


Figure 14.1

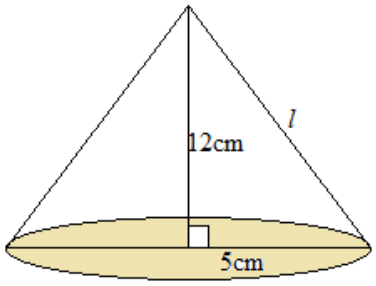
From the diagrams the **Curved surface area** $= \boldsymbol{\pi rl}$ and the area of a **circle** $= \boldsymbol{\pi r^2}$, therefore

The total surface area of a **cone**$= \pi rl + \pi r^2 = \pi r(l + r)$

Worked Example [4]

Question

In the diagram below is a cone with a missing slant height l. Take π to be $\frac{22}{7}$.



Find,

a) the slant height l

b) the base area

c) the curved surface area

d) the total surface area

Solutions

(a) $l^2 = 5^2 + 12^2$

$l^2 = 25 + 144$

$l^2 = 169$

$l = \sqrt{169}$

$l = 13$

(b) Base area $= \pi r^2$

Base area $= \frac{22}{7} \times 5^2 = 78\frac{4}{7}$ cm^2

(c) Curved surface area$= \pi rl = \frac{22}{7} \times 5 \times 13 = 204\frac{2}{7}$ cm^2

(d) Total surface area $= 78\frac{4}{7} + 204\frac{2}{7} = 282\frac{6}{7}$cm^2

14.2.2 Total surface area of a cylinder

It is also important that we explore the surface area of a cylinder by reducing the curved part to simple rectangle as shown in Figure 14.2 below.

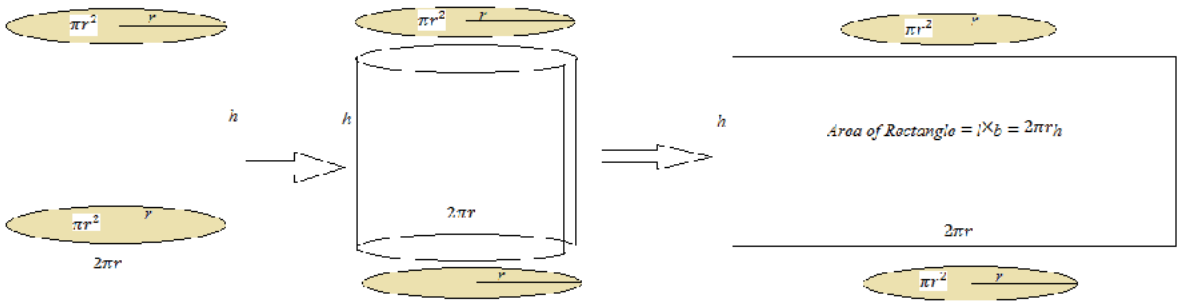


Figure 14.2

It can be deduced from the diagrams above that the curved surface area of a **cylinder** is $\mathbf{2\pi rh}$ and the area of two **circles** is $\mathbf{2\times \pi r^2}$, therefore,

The total surface area of **cylinder** $= 2\pi r^2 + 2\pi rh = 2\pi r(r + h)$

Worked Example [5]

Questions

A cylinder has a base radius of 21cm and a perpendicular height of 35cm. Take π to be 3.142. Find

(a) the curved surface area of the cylinder

(b) the total surface area

Solutions

(a) Curved surface area of cylinder = $2\pi rh$

Curved surface area of cylinder = $2 \times 3.142 \times 21 \times 35$ (a calculator may be used)

$= 4618.74\text{cm}^2$

(b) The total surface area of cylinder $= 2\pi r(r + h)$

$= 2 \times 21(21 + 35)$

$= 2352\text{cm}^2$

14.2.3 Surface area of a Sphere

A sphere is ball in shape. In figure 14.3 below, we have given a sphere and a hemisphere (half of a sphere).

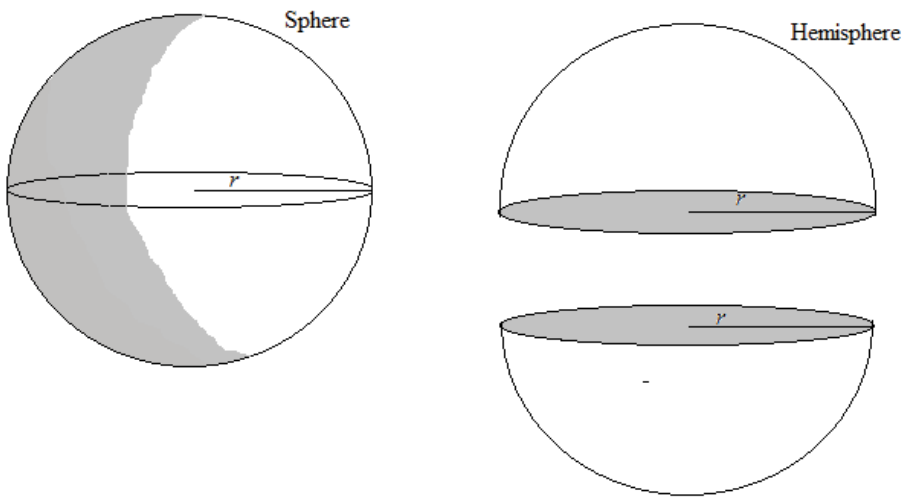


Figure 14.3

The surface area of a sphere $= 4\pi r^2$

Note that the surface area of hemisphere is half the surface area of sphere plus area of a circle as shown in figure 14.3 above.

Therefore, the surface area of a hemisphere $= 2\pi r^2 + \pi r^2 = 3\,\pi r^2$

Note that proof of these formulae is beyond the scope of this module.

Worked Example [6]

Question

A plastic ball has a radius of 5.6cm. Take π to be $\frac{22}{7}$. Find the surface area of the ball, correct to the nearest cm^2

Solution

The surface area of a sphere $= 4\pi r^2$

The surface area of a sphere $= 4 \times \frac{22}{7} \times 5.6 \times 5.6$

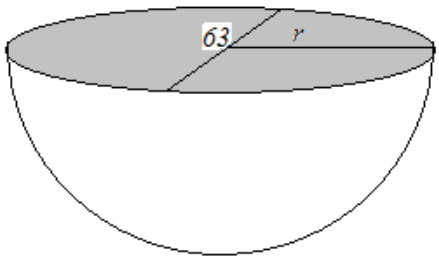
$=394.24$

$=394\ cm^2$

Worked Example [7]

Question

The diagram below shows a hemisphere with a diameter of 63m. Take π to be 3.142. Find the total surface area of the hemisphere correct to 3 significant figures.



TIP

- Radius is the only property of a circles that is need to obtain a surface area

Solution

The surface area of a hemisphere $= 3\ \pi r^2$

$$= 3 \times 3.142 \times \left(\frac{63}{2}\right)^2$$

$$= 9352.9485$$

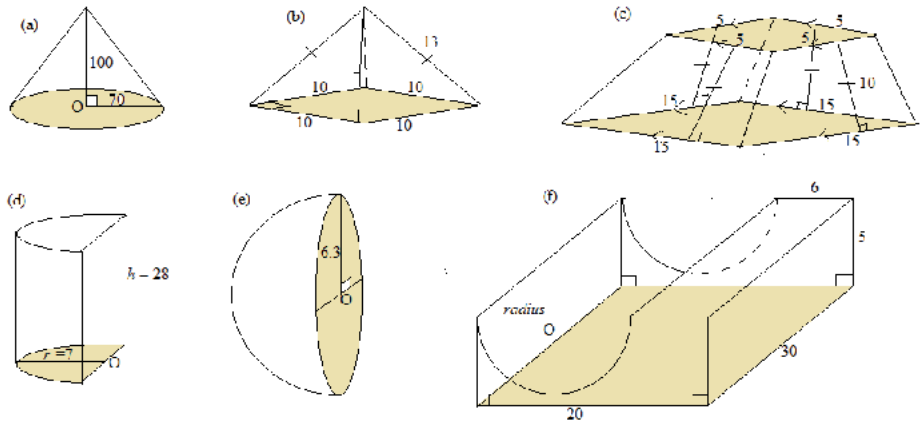
$$= 9350\text{m}^2 \qquad \text{3s.f}$$

We have tried our best to provide various examples on surface area. In order to grasp there various concepts, you need some practice. Attempt the following activity.

Activity (*14.1*) Surface area of solid shapes

Questions

1. Calculate the total surface of each shape below. All dimensions are in mm. Use the value $3\frac{1}{7}$ forπ. O represents the centre of a circle. Give your answers correct to 3. s.f



Answers

a) 42300

b) 340

c) 350

d) 1162

e) 374

f) 1786

14.3 VOLUME OF PRISMS

Prisms come in various shapes and it is crucial to refer to the definition of the a prism so that you identify the prisms correctly before you attempt any question. In Unit 13 and also in Level 1, you covered simplified volumes of cones, cuboids and cylinders. Now we need to cover the aspect on volumes of prisms. The general formula for the volume of a prism is given below.

$$\textbf{Volume of a prism} = \textbf{Base Area} \times \textbf{Perpendicular Height}$$

In the diagrams in fig 14.4 are some of the prisms in their various forms, where the shaded part represents the **base area** and **h** represents the perpendicular height

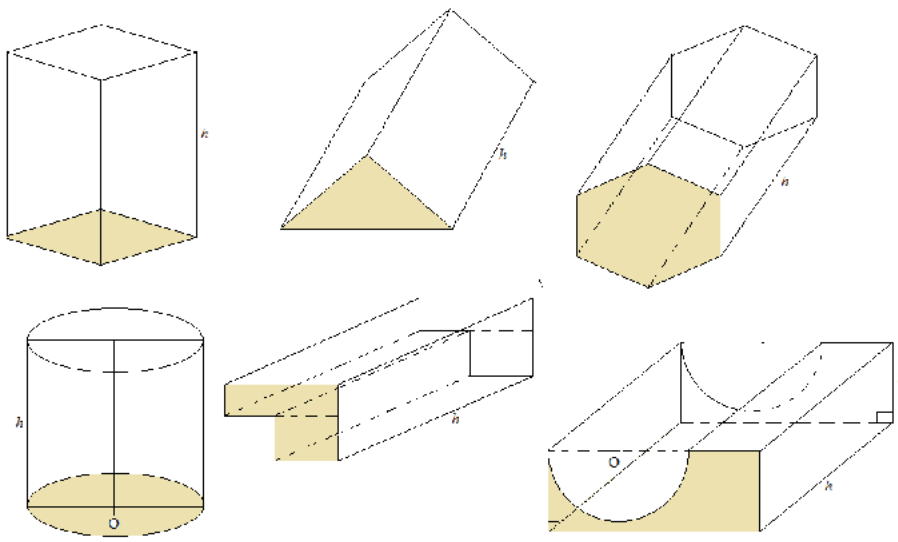
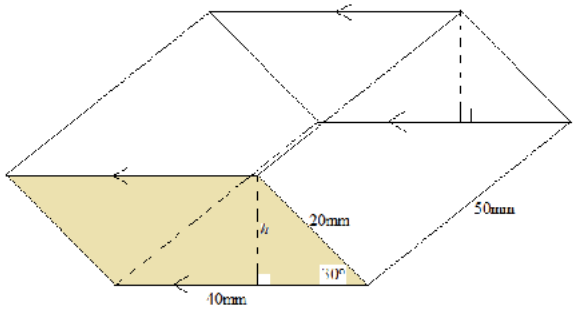


Figure 14.4

Worked Example [8]

Questions

The diagram below shows a parallelogram based wooden prism.



Find

(a) the volume of the wooden prism

(b) density(g/cm^3) of the wooden prism if its mass is 2kgs

Solutions

Volume of a prism = Base Area × Perpendicular Height

Remember (Area of parallelogram = $ab\sin\theta$)

(a) **Volume of a prism** $= 40 \times 20 \sin 30° \times 50$

$= 40 \times 10 \times 50$

$= 20000\text{mm}^3$

(b) Convert the quantities to the required units, that is (g/cm^3)

Mass 2kg $= 2000\text{g}$

Volume $= \frac{20000}{10\times10\times10}\text{cm}^3 = 20\text{cm}^3$

Density $= \frac{\text{Mass}}{\text{Volume}}$

Density $= \frac{2000\text{g}}{20\text{cm}^3}$

Density $= 100\text{g/cm}^3$

14.3.1 Capacity

It very crucial that we include the aspect of capacity. Table 14.1 below illustrates how some of the units are converted in capacity problems.

Table 14.1 - Conversion table

Volume in cubic units	Capacity in litres
1000 cm^3	1 litre
1m^3	1000 litres

Worked Example [9]

Question

5 litres of petrol are poured into a Jeri-can container whose cross- section is a square of sides 20cm. How deep is the Jeri-can container?

Solution

Volume = base area × perpendicular height

$$5 \times 1000 = 20 \times 20 \times h$$

$$h = \frac{5 \times 1000}{20 \times 20}$$

$$h = \frac{5000}{400}$$

$$h = 12.5$$

Now that we have gone through calculations related to volume of prisms, attempt the following activity.

Activity (*14.2*) Volumes of prisms and capacity

Questions

1) A truck carries rectangular tank of molasses which measures 23m long, 1.8m wide and 1,2m high. Calculate the volume of the contents in m^3
2) Triangular based metallic prism has a volume of $20cm^3$. It is melted and cast into a rectangular prism of height 4cm. If the rectangular prism has a square base find the area of the square base.
3) Find the volume of each of the following prisms. All dimensions are given in cm. Leave your answers correct to 3 significant figure where necessary

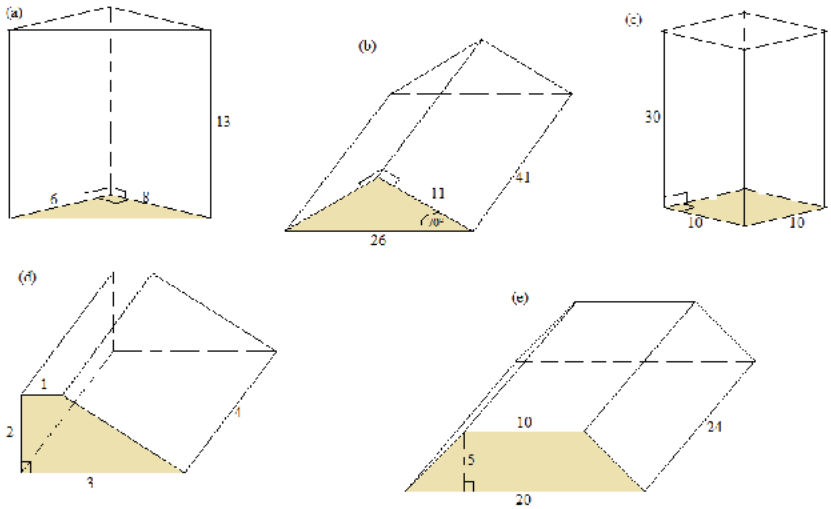


Figure 14.11

4) A cylindrical tank of water is full and has a diameter of 1.2m and a height of 1.4m. The water is poured into a rectangular tank 1.6m long and 1.1m wide. What is the depth of the rectangular tank?
5) An oil drum is 120cm square and 135cm deep. It is full of oil. How many times can a 18-litre Jeri-can can be filled from the drum

Answers

1) 49.68
2) 5 ¾
3) a) 624 b) 11726 c) 3000 d) 16 e) 1800
4) 0.9m
5) 108

14.3.2 Volume of a sphere

We have a sphere under surface areas and defined it as ball shaped. In figure 14.3 below, we have given a sphere and a hemisphere (half of a sphere).

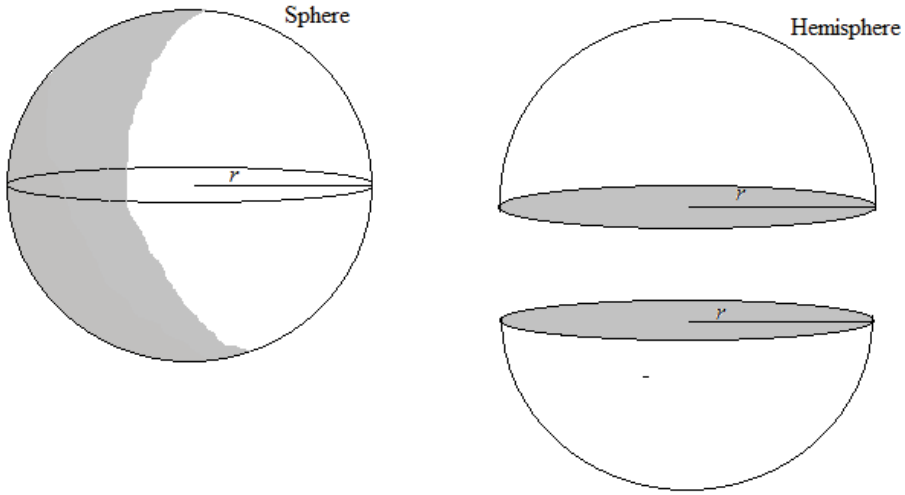


Figure 14.3

The Volume of a sphere $= \frac{4}{3}\pi r^3$

Note that the volume of a hemisphere is half the volume of sphere.

Therefore, the volume of a hemisphere $= \frac{2}{3}\pi r^3$

Note that proof of these formulae is beyond the scope of this module.

Worked Example [10]

Question

A metal ball 12m in diameter is melted and cast into small balls of 1m in diameter. How many of the small balls will there be? Use the value $3\frac{1}{7}$ for π. Give your answer correct to nearest unit.

Solution

TIP:

- We need to know the volume of the bigger ball and the volume of the smaller ball first then divide smaller volume into the bigger one.

Volume of big ball= $\frac{4}{3} \times \frac{22}{7} \times 6^3$

$$= \frac{4}{3} \times \frac{22}{7} \times 6 \times 6 \times 6$$

$= 905\frac{1}{7}\text{m}^3$

Volume of small ball $= \frac{4}{3} \times \frac{22}{7} \times (\frac{1}{2})^3$

$$= \frac{4}{3} \times \frac{22}{7} \times \frac{1}{8}$$

$= \frac{11}{21}\text{ m}^3$

Number of small balls $= \frac{\text{volume of big ball}}{\text{volume of small ball}}$

14.4 VOLUME OF FRUSTUMS

A frustum has been defined under key terms and below are some of the examples of frustums

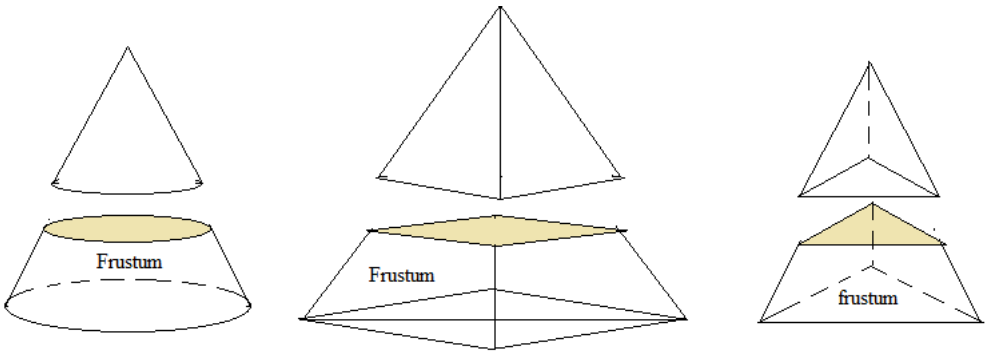


Figure 14.4

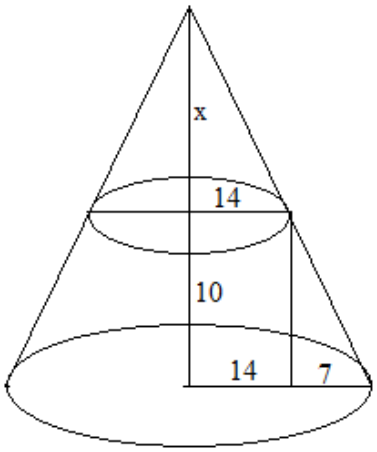
Worked Example [11]

Question

A frustum of cone has bottom diameter of 42cm and a top diameter of 28cm respectively and a depth of 10cm. find the volume of the frustum. Take π to be $\frac{22}{7}$.

Solution

A sketch of the diagram is necessary since ratios will be considered in finding the total altitude of the cone



TIP: Equate similar triangles as follows

$$\frac{x}{14} = \frac{10}{7}$$

$$x = 20$$

Therefore, the total perpendicular height is 30

<u>Remember</u> (Volume of a cone $= \frac{1}{3}\boldsymbol{\pi r^2 h}$**)**

Volume of a cone $= \frac{1}{3} \times \frac{22}{7} \times 21 \times 30$

$= 13860$

Activity (*14.3*) Volume of spheres and frustums

Questions

Use the value 3.142 for π

1) Calculate the volume to 3 significant figures of each of the following.
   a) A hemisphere of radius 3.5mm
   b) A hemisphere of diameter 18cm
   c) A sphere of radius 13 m
   d) A sphere of diameter 29 mm
2) A pyramid on a base of 100cm square is 150cm high
   a) Find the volume of the pyramid.
   b) If the top height of 60cm of the pyramid is removed, find the volume of the remaining frustum.
3) Find the capacity in litres of a builders tin 24cm in diameter at the top, 16 cm in diameter at the bottom and 20cm deep
4) Find in m^3, the volume of a frustum of a cone which has the top and bottom diameter of 3m and 2m respectively and a depth of 1m

Answers

1) a) 89.8 b)1527.4 c) 9 206.5 d) 102 201.5
2) a) 500 000 b) 180 000
3) 2026.7π
4) 1.58π

14.5 AREA AND VOLUME OF SOLID SHAPES (COMPOSITE AND HOLLOW SHAPES)

Under this concept we are going to look at the combined solid shapes. Solid shapes make up most tools, equipment and utensils that we use in everyday life. Think of pencils, pots, salt shakers, bottles etc. We are therefore going to find surface areas, exposed surface area, volumes of combine and removed solid shapes. Given below are a few examples of combined solid shapes.

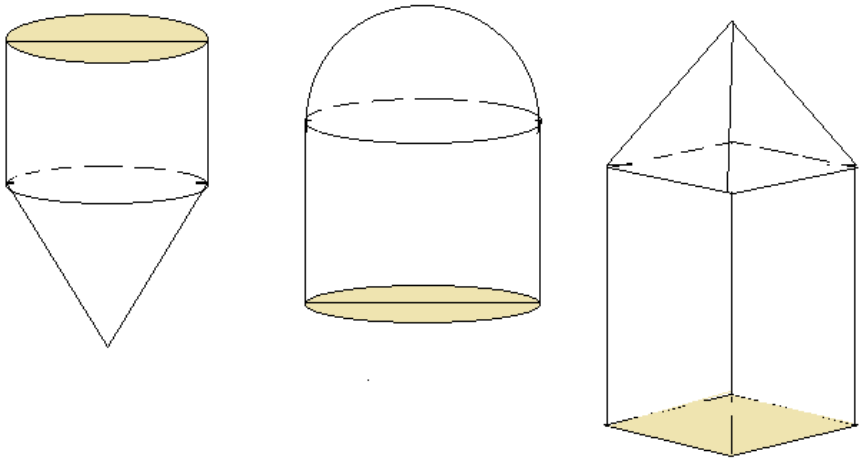


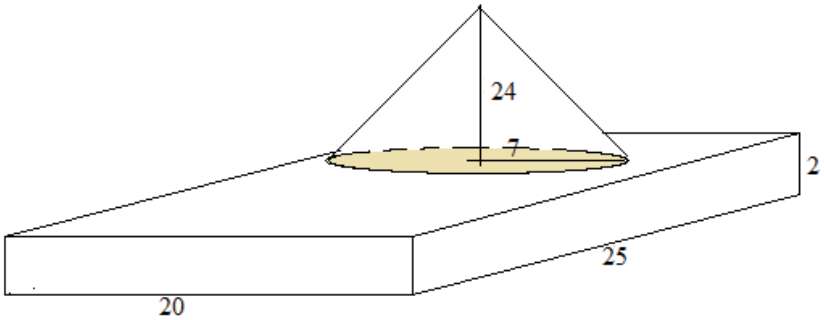
Figure 14.6

14.5.1 Surface and exposed surface areas of combined solid shapes

Worked Example [12]

Questions

The diagram below is wooden cone placed flat on a wooden block. All dimensions are given in cm. Take π to be $\frac{22}{7}$. Give your answers correct to 3.s.f.



Calculate

(a) the curved surface area of the cone

(b) the exposed area of the whole shape

Solutions

<u>Remember (</u>Curved surface area $= \pi rl$)

(a) Slant height (l)

$l^2 = 7^2 + 24^2$

$l = 25$

Curved surface area $= \frac{22}{7} \times 7 \times 25$

$= 550 \text{ cm}^2$

(b) The exposed surface area is obtained by removing the area where the two combined shapes are in contact

In this case we are going to add the surface area of the prism and the curved surface are of the cone minus the area of a circle, which is the area where the two shapes are in contact.

$$\text{Exposed surface area} = 2lb + 2lh + 2bh + \pi rl - \pi r^2$$

$$= 2 \times 25 \times 20 + 2 \times 25 \times 2 + 2 \times 20 \times 2 + 550 - \frac{22}{7} \times 7^2$$

$$= 1000 + 100 + 80 + 550 - 154$$

$$= 1576 \text{ cm}^2$$

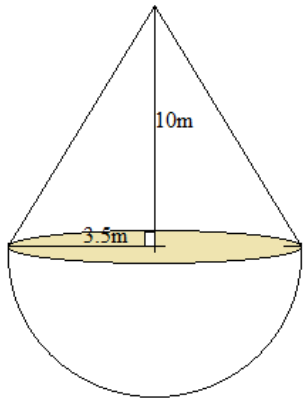
14.5.2 Composite solids

This refers to addition of volumes solids

Worked Example [13]

Questions

In the diagram below is a combination of a hemisphere and a cone. Find the volume of the solid shape in terms of π



Solutions

<u>Remember (</u>Volume of cone $= \frac{1}{3}\pi r^2 h$ and Hemisphere $= \frac{2}{3}\pi r^3$)

Volume of the Cone $= \frac{1}{3} \times \pi \times 3.5^2 \times 10 = 40\frac{5}{6}\pi$ m^3

Volume of the hemisphere $= \frac{2}{3} \times \pi \times 3.5^3 = 28\frac{7}{12}\pi$ m^3

Total volume of the solid $= 40\frac{5}{6}\pi + 28\frac{7}{12}\pi = 69\frac{5}{12}\pi$ m^3

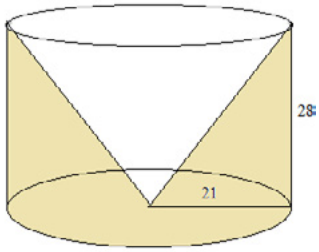
14.5.3 Hollow solids (shapes)

This concept involves subtraction of solids.

Worked Example [14]

Question

In the diagram below is a cylinder with a cone placed upside down such the liquid in the cylinder occupies the shaded part only. All dimensions are given in cm. Take π to be $\frac{22}{7}$. Find the capacity of the liquid to the nearest litre



Solution

<u>Remember (</u>Volume of cone $= \frac{1}{3}\pi r^2 h$ and cylinder $= \pi r^2 h$)

To find the volume of the liquid we simply subtract the volume of the cone from that of a cylinder

Volume of cone $= \frac{1}{3} \times \frac{22}{7} \times 14^2 \times 28 = 5749\frac{1}{3}$ cm^3

Volume of cylinder $= \frac{22}{7} \times 14^2 \times 28 = 17248$ cm^3

Total Volume $= 17248 - 5749\frac{1}{3} = 11498.6667 = 11498.7$cm^3

Capacity in litres $= \frac{11498.7}{1000} = 11.4987 = 11$ ltrs

Activity (*14.4*) Area and volume of shapes

Questions

1) A rectangular tin, 1.8m by 1.2m by 0.6m, contains four identical cubical metals of side 0.6m each. Calculate the percentage to the nearest unit of the volume of the tin occupied by the cubical metals
2) Calculate the volume in m^3 of the material that make up a cylindrical conduit pipe 18m long, the internal and external diameters big 1,6cm and 1,8cm respectively
3) A solid ball of radius 3.5m is dropped into cylindrical tank of radius 14m. Calculate
   (a) the volume of the ball
   (b) the volume of the tank in terms of h(the height)
   (c) the raise in water level if the initial depth was 10m
4) Find volumes of the following shapes. Take3.142 for π. Leave your answers correct to 3.s.f

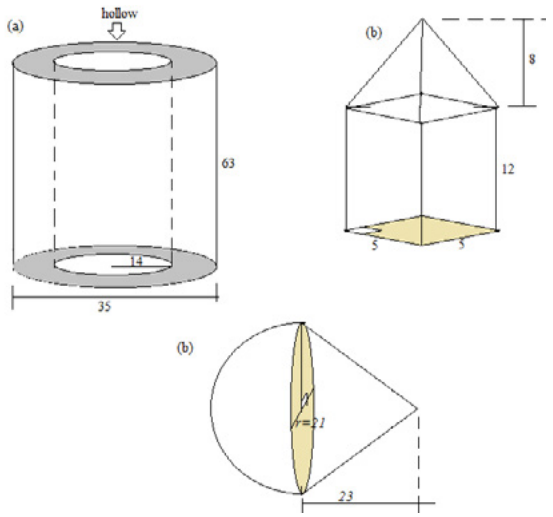


Figure 14.20

REFLECTION

- Solution for volume will always depend on the base area of a plane shape.
- Frustums are extracts of cones and pyramids
- Composite solids is addition of solids while hollow solids are subtraction of solids

- Volume in cubic units can be converted to capacity in litres for liquids
- Formulae for surfaces areas of cylinders and cones can be simplified into ordinary plane shapes

14.6 SUMMARY

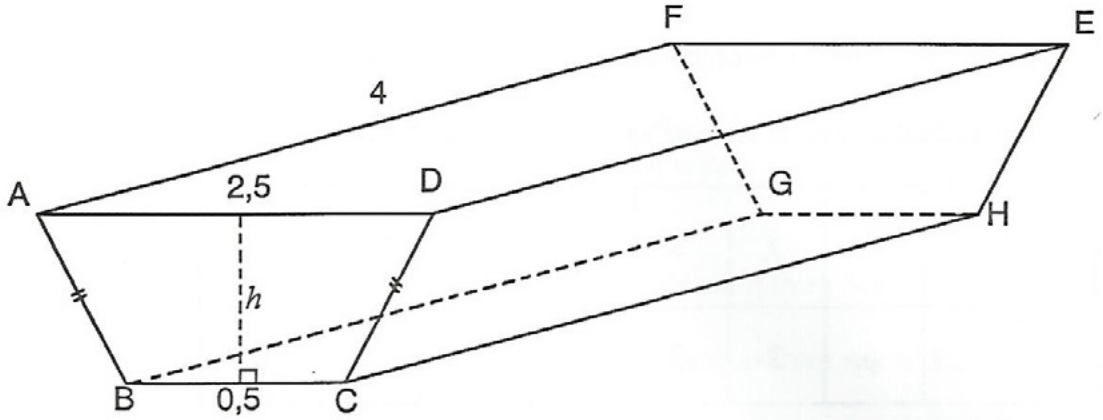
This unit has marked the end of measure and Mensuration 2 where areas and volumes of solids have been explored in detail. The unit has made it clear that composite and hollow shapes are as result of addition and subtraction respectively. The concept on capacity has also been factored in and some notes and examples on conversion of capacity were also given.

14.7 Further Reading

1. Macrae, M, Madungwe, L and Mutangadura, A (2017) New General Mathematics Book 3, Pearson Capetown.
2. Macral, M, Madungwe, L and Mutangadura, A (2017) New General Mathematics Book 4, Pearson Capetown.
3. Meyers, C; Graham, B; Dawe, L (2004) Mathscape Working Mathematically, 9th Ed, MacMillan
4. Pimentel R and Wall T (2011) International Mathematics, Hodder UK
5. Rayner D (2005) Extended Mathematics. Oxford New York

14.8 Assessment Test

1. June 2003 QN12



In the diagram, ABCDEFGH is a water trough with base BCHG. The cross-section ABCD is an isosceles trapezium with AD parallel to BC, AD = 2,5m, BC = 0,5m and AB = DC. The trough is 4m long.

(a) Given that the area of the trapezium, ABCD, is $3m^2$, calculate

(i) the height h, [2]

(ii) the length of AB. [2]

(b) The interior of the trough was painted at a cost of \$70 per m^2. Calculate the cost of painting the interior. [3]

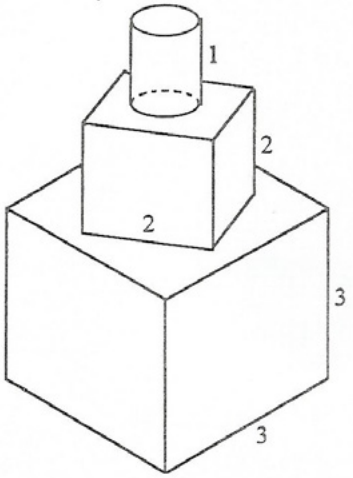
(c) Find the capacity of the trough giving the answer in litres. [2]

(d) **In this part of the question take π to be $\frac{22}{7}$.**

The trough was filled with water from a cylindrical reservoir of base radius 5m. Calculate the drop in the water level of the reservoir giving the answer in centimetres. [3]

2. Nov 2003 QN 7

In this question take π to be $\frac{22}{7}$.



A wooden sculpture consists of two cubes and a cylinder as shown in the diagram above. The larger cube sits flat on the ground and is of side 3 metres. The smaller cube is of side 2 metres and the cylinder has radius 0,7 metres and a height of 1 metre.

(a) Calculate the exposed surface area of

(i) the cylinder,

(ii) the 2-metre cube,

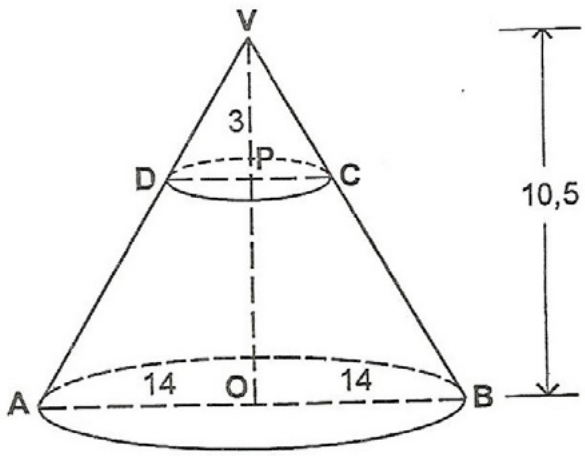
(iii) the 3-metre cube. [6]

(b) The exposed surfaces are to be painted. One 2-litre tin of paint is needed to paint one square metre of the sculpture.

Calculate the number of tins to be bought. [3]

(c) Each 2-litre tin costs \$1 473. Calculate the total cost of paint to be bought. [3]

3. Nov 2003 QN 12 paper1



In this question take π to be $\frac{22}{7}$.

A large solid cone is cut into two parts by a plane parallel to its horizontal circular base, the upper part being a small solid cone.

The figure above represents the solid, with V the vertex, AB the base diameter and VO the perpendicular height of the large cone. DC is the base diameter and VP the perpendicular height of the smaller cone.

Given that AO = OB = 14 cm, VO = 10,5 cm and VP = 3 cm,

calculate

(a) DP, [3]

(b) DA, [3]

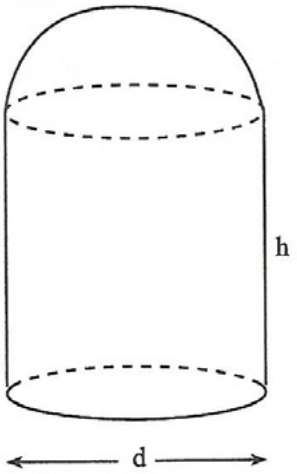
(c) the ratio of the volume of the smaller cone to the volume of the larger cone, giving the answer as a fraction in its lowest terms. [3]

(d) the curved surface area of the smaller cone. [3]

[Volume of a cone = $\frac{1}{3}\pi r^2 h$. Curved surface area of a cone = $\pi r \ell$.]

4. Nov 2013 QN 3(c)

A salt shaker is made up of a cylinder of height h and a hemisphere of internal diameter d as shown below.



(i) Write down an expression for the volume of the salt shaker in terms of π, d and h.

(ii) Find the internal volume of the salt shaker if $h = 11$ cm and $d = 3{,}5$ cm, leaving the answer in terms of π.

$$\left[\text{Volume of sphere} = \frac{4\pi r^3}{3}\right]$$ [5]

UNIT 15 - INDICES AND LOGARITHMS

CONTENTS

15.1 INTRODUCTION

Do you know that we can write numbers in form of indices or logarithms? Sometimes a number may be too large that you might end up writing it in index form or as a logarithmic function. In this unit we are going to evaluate indices and logarithms as well as solve equations involving indices and logarithms.

OBJECTIVES

After going through this unit, you should be able to:

- state the laws of indices
- evaluate indices
- solve equations on indices
- state the laws of logarithms
- simplify logarithms
- solve equations on logarithms

KEY WORDS

Index – is the number in the form x^a where x is the base and a is the power. The plural for index is indices.

Logarithm - a logarithm is a power

10 hours

You are advised not to spend more than 10 hours on this unit.

STUDY SKILLS

The key skill to mastery of mathematical concepts is practice. You need to solve as many problems on Indices and Logarithms as possible for you to grasp all the concepts in this topic.

15.2 INDICES

The term index refers to the power to which a number is raised. For example, 3^2 means that the number 3 is raised to the power of 2. The number 2 is known as the index. Indices is the plural of index. It means we are multiplying 3 by itself 2 times.

$3^2 = 3 \times 3$

$=9$

Worked Example [1]

Questions

Simplify

a) 2^4

b) -3^3

Solutions

a) 2^4, means that 2 is multiplied by itself 4 times which is;

$2 \times 2 \times 2 \times 2 = 16$

b) -3^3, the question is the same as in (a) above that is

$-3 \times -3 \times -3 = -27$

15.2.1 Evaluation

In order for us to evaluate indices there are laws which should be used. These laws are as follows:

1) $a^x \times a^y = a^{x+y}$.
2) $a^x \div a^y = a^{x-y}$
3) $a^0 = 1$.
4) $a^{-b} = \frac{1}{a^b}$
5) $(a^b)^c = a^{b\times c}$.
6) $a^{\frac{1}{x}} = \sqrt[x]{a}$.
7) $a^{\frac{y}{x}} = \sqrt[x]{a^y}$ or $(\sqrt[x]{a})^y$
8) $(\frac{a}{b})^{-(\frac{y}{x})} = (\frac{b}{a})^{\frac{y}{x}}$ or $\frac{1}{(\frac{a}{b})^{\frac{y}{x}}}$ or $\frac{1}{\sqrt[x]{(\frac{a}{b})^y}}$

Worked Example [2]

Questions

By making use of the above laws evaluate the following questions;

a) $5x^2 \times 3x^3$
b) $25y^2 \div 5y^3$
c) $8^0 + 8^2$
d) 3^{-2}
e) $(2^3)^2$

Solutions

a) In this question, 5 and 3 are different numbers so we have to treat them as usual

$5x^2 \times 3x^3 = 5 \times 3 \times x^2 \times x^3$

$= 15x^{2+3}$

$= 15x^5$

b) $25y^2 \div 5y^3 = \frac{25}{5} \times y^2 \div y^3$

$= 5y^{2-3}$

$= 5y^{-1} = 5 \times \frac{1}{y}$

$= \frac{5}{y}$

c) $8^0 + 8^2 = 1 + 64$, since there is no (×) or (÷), we don't apply laws of indices

$= 65$

d) $3^{-2} = \frac{1}{3^2}$

$= \frac{1}{9}$.

e) $(2^3)^2 = 2^{3\times 2}$

$= 2^6$,

$= 64$.

Worked Example [3]

Questions

Simplify the following fractional indices

a) $27^{-\frac{1}{3}}$

b) $6^{\frac{1}{4}} \times 6^{\frac{3}{4}}$

c) $(\frac{4}{9})^{-2}$

d) $(\frac{18}{50})^{\frac{1}{2}}$

e) $0{,}125^{-\frac{2}{3}}$

f) $81^{\frac{1}{4}} \times 0{,}064^{\frac{1}{3}}$

Solution

a) $27^{-\frac{1}{3}}$

$= \left(\frac{1}{27}\right)^{\frac{1}{3}}$

$= \sqrt[3]{\frac{1}{27}}$

$= \frac{1}{3}$

b) $6^{\frac{1}{4}} \times 6^{\frac{3}{4}}$

$= 6^{\frac{1+3}{4}}$

$= 6^1$

$= 6$

c) $(\frac{4}{9})^{-2}$

$= \left(\frac{9}{4}\right)^2$

$= \frac{81}{16}$

$= 5\frac{1}{16}$

d) $(\frac{18}{50})^{\frac{1}{2}}$

$= \left(\frac{9}{25}\right)^{\frac{1}{2}}$

$= \frac{3}{5}$

e) $0{,}125^{-\frac{2}{3}}$

$= (\frac{125}{1000})^{-\frac{2}{3}}$

$= (\frac{1000}{125})^{\frac{2}{3}}$

$= (\sqrt[3]{\frac{1000}{125}})^2$

$= (\frac{10}{5})^2$

$= 2^2$

$= 4$

f) $81^{\frac{1}{4}} \times 0{,}064^{\frac{1}{3}}$

$= \sqrt[4]{81} \times \left(\frac{64}{1000}\right)^{\frac{1}{3}}$

$= 3 \times \sqrt[3]{\frac{64}{1000}}$

$= 3 \times \frac{4}{10}$

$= \frac{12}{10}$

$= 1\frac{1}{5}$

Now that we have looked at some examples where indices are simplified, I think you are now prepared to attempt the following questions.

Activity (*15.1*) Laws of indices

Questions

Simplify the following expressions

1) 3^{-2}
2) $(4x)^3 \times 2x^2$
3) $27^{\frac{2}{3}}$
4) $\left(1\frac{9}{16}\right)^{-\frac{3}{2}}$
5) $4^x \times 4^{-x}$
6) $0{,}216^{-\frac{2}{3}}$
7) $\left(\frac{27}{48}\right)^{\frac{-5}{2}}$
8) $5^{-2} \div 16^{\frac{3}{4}}$
9) $2y^3 \div 4y^{-4}$
10) $\sqrt[3]{27x^6}$

Answers

1) $\frac{1}{9}$
2) $128x^5$
3) 9
4) $\frac{64}{125}$
5) 1
6) $2\frac{7}{9}$
7) $\frac{1024}{243}$
8) $\frac{1}{200}$
9) $\frac{1}{2}y^7$
10) $3x^2$

15.2.2 EQUATIONS

There are 2 types of equations involving indices which are

- Type 1:
  It is an equation whereby the unknown letter is the power of a number for instance ;
  $2^x = 16.$

- Type 2:
  It is an equation whereby the unknown letter is the base and an index is a number for instance;
  $x^3 = 64.$

Worked Example [4]

Question

Solve for x

$2^x = 16.$

Solution

First we have to express 16 as powers of 2 like this;

$2^x = 2^4$, now the bases are the same so we equate the powers

$\therefore x = 4$

Worked Example [5]

Question

Solve $8^x = \frac{1}{64}$

Solution

We again express $\frac{1}{64}$ as powers of 8

$8^x = \frac{1}{8^2}$,

$8^x = 8^{-2}$, since the bases are the same we equate the powers

$\therefore x = -2$

Worked Example [6]

Question

Find the value of x in $4^{x+1} = 32$

Solution

We express 4 and 32 as powers of the common base that is 2

$2^{2(x+1)} = 2^5$

$2^{2x+2} = 2^5$, since the bases are the same we equate the powers

$2x + 2 = 5$, solving the equation we get;

$x = 1\frac{1}{2}$

Worked Example [7]

Questions

Solve $3^{2(m-3)} \times 3^{5m} = 27$

Solution

By applying the laws of indices on the L.H.S and expressing 27 as powers of 3 we get;

$3^{2(m-3)+5m} = 3^3$, equating the powers we get;

$2(m - 3) + 5m = 3$

$2m - 6 + 5m = 3$

$7m = 3 + 6$

$7m = 9$, dividing throughout by 7 we get

$m = 1\frac{2}{7}$

Worked Example [8]

Question

Given that $5^{-2x} - 5^2 = 100$ find the value of x.

Solution

Since there is a subtraction sign between the 2 indices on the L.H.S we regroup the equation

$5^{-2x} = 100 + 5^2$

$5^{-2x} = 100 + 25$

$5^{-2x} = 125$, expressing 125 as powers of 5 we get;

$5^{-2x} = 5^3$, equating the powers we get

$-2x = 3$, dividing both sides by -2

$x = -1\frac{1}{2}$

Worked Example [9]

Question

Solve the equation

$x^3 = 64.$

Solution

In this equation x is now the base, so we have to find the inverse of its power

$x^3 = 64$, the inverse of 3 is $\frac{1}{3}$ so raise every term in the equation to the power $\frac{1}{3}$

$(x^3)^{\frac{1}{3}} = 64^{\frac{1}{3}}$, simplifying we get;

$x = \sqrt[3]{64}$,

$\therefore x = 4$

Worked Example [10]

Question

Solve the equation

$y^{-\frac{1}{4}} = 3.$

Solution

The reciprocal/inverse of $-\frac{1}{4}$ is -4 so we raise every term in the equation to the power -4

$(y^{-\frac{1}{4}})^{-4} = 3^{-4}$, simplifying we get,

$$y = \frac{1}{3^4}$$

$$\therefore y = \frac{1}{81}$$

Worked Example [11]

Questions

Solve for d in $d^{\frac{2}{3}} = 16$.

Solution

The inverse of $\frac{2}{3}$ is $\frac{3}{2}$

$(d^{\frac{2}{3}})^{\frac{3}{2}} = 16^{\frac{3}{2}}$, simplifying we get

$$d = \sqrt{16^3}$$

$$d = 4^3$$

$$\therefore d = 64$$

Worked Example [12]

Questions

Solve for x

$$6x^{\frac{1}{2}} = -18x.$$

Solution

First we divide both sides by 6 to get

$x^{\frac{1}{2}} = -3x$, dividing both sides by x we get;

$x^{\frac{1}{2}} \div x = -3$, applying the laws of indices on the L.H.S we get;

$x^{\frac{1}{2}-1} = -3$,

$x^{-\frac{1}{2}} = -3$, multiplying the powers by -2 (the inverse of $\frac{-1}{2}$)

$x = (-3)^{-2}$

$$x = \frac{1}{(-3)^2}$$

$$\therefore x = \frac{1}{9}$$

Activity (*15.2*) Equations involving powers

Questions

Solve the following equations

1) $x^{-\frac{1}{2}} = 3$
2) $4y^3 = 128$
3) $x^{-\frac{2}{3}} = 25$
4) $5m^{-1} = -10$
5) $3n^{\frac{1}{3}} = 243$
6) $9^{x+1} = 27$
7) $5^{3-2x} = 625$
8) $5^{2(x-3)} \div 5^{5x} = 125$
9) $4^{3y} \div \frac{1}{64} = 4^{y(y+1)}$
10) $3^{m-2} = \frac{1}{9}$

Answers

1) $\frac{1}{9}$ 2) 2 3) $\frac{1}{125}$ 4) $-\frac{1}{2}$

5) 729 6) $\frac{1}{2}$ 7) $\frac{1}{2}$ 8) –3

9) or –1 10) 0

15.3 LOGARITHMS

Have you asked yourself what a logarithm is and how does it work? Ok, for you to know what a logarithm is, you have to know what an index is. A logarithm is a power; let's take $10^3 = 1000$, but $\log_{10} 1000 = 3$.

15.3.1 Theory of logarithms

There are 3 major laws of which are ;

1) $\log_x MN = \log_x M + \log_x N$
2) $\log_x \frac{M}{N} = \log_x M - \log_x N$
3) $\log_x (M)^p = p \log_x M$

Let us make use of the above laws in the following examples

Worked Example [13]

Questions

Given that $\log_{10} 3 = 0{,}47712$ $\log_{10} 2 = 0{,}30103$ $\log_{10} 5 = 0{,}6691$

Calculate

a) $\log_{10} 15$
b) $\log_{10} 20$
c) $\log_{10} 25$
d) $\log_{10} 1\frac{2}{3}$
e) $\log_{10} 0{,}03$
f) $\log_{10} 32$

Solutions

a) $\log_{10} 15$
$= \log_{10} (3 \times 5)$
$= \log_{10} 3 + \log_{10} 5$
$= 0{,}47712 + 0{,}6691$
$= 1{,}14622$

b) $\log_{10} 20$
$= \log_{10}(2 \times 10)$
$= \log_{10} 2 + \log_{10} 10$
$= 0{,}30103 + 1$
$= 1{,}30103.$

c) $\log_{10} 25$
$= \log_{10} 5^2$
$= 2 \log_{10} 5,$
$= 2 \times 0{,}47712,$
$= 0{,}95424$

d) $\text{Log}_{10} 1\frac{2}{3}$
$= \log_{10} \left(\frac{5}{3}\right)$
$= \log_{10} 5 - \log_{10} 3$
$= 0{,}6691 - 0{,}47712$
$= 0{,}19198$

e) $\log_{10} 0{,}03$

$= \log_{10}\left(\frac{3}{100}\right)$

$= \log_{10} 3 - \log_{10} 100$

$= 0{,}47712 - 2$

$= -1{,}52288$

f) $\log_{10} 32$

$= \log_{10} 2^5$

$= 5\log_{10} 2$

$= 5 \times 0{,}30301$

$= 1{,}51505$

Worked Example [14]

Questions

Given that $\log x = 3$ $\quad \log y = -2$ $\quad \log z = \frac{3}{4}$

Find

a) $\log xy$

b) $\log \frac{x}{y}$

c) $\log y^{-2}$

d) $\log \sqrt{z}$

e) $\log \frac{1}{z}$

Solutions

a) $\log xy = \log x + \log y$

$= 3 - 2$

$= 1$

b) $\log \frac{x}{y} = \log x - \log y$

$= 3 - (-2)$

$= 5$

c) $\log y^{-2} = -2\log y$

$= -2 \times -2$

$= 4$

d) $\log \sqrt{z} = \log z^{\frac{1}{2}}$

$= \frac{1}{2}\log z$

$= \frac{1}{2} \times \frac{3}{4}$

$= \frac{3}{8}$

e) $\log \frac{1}{z} = \log 1 - \log z$

$= 0 - \frac{3}{4}$

$= -\frac{3}{4}$

Activity (*15.3*) Logarithms

Questions

Given that $\log 2 = 0{,}301$, $\log 3 = 0{,}477$, $\log 5 = 0{,}699$ and $\log 7 = 0{,}845$

Find

a) $\log 4$
b) $\log 27$
c) $\log 14$
d) $\log 35$
e) $\log 49$
f) $\log \frac{2}{3}$
g) $\log 1\frac{2}{5}$
h) $\log 0{,}016$
i) $\log 300$
j) $\log 490$
k) $\log \sqrt[5]{7}$
l) $\log 2\frac{1}{3}$
m) $\log 4\frac{17}{27}$

Evaluation of logarithms

We are now going to look at how logarithms are evaluated. Consider the following example.

Worked Example [15]

Questions

Simplify as far as possible

a) $\frac{\log 16}{\log 4}$
b) $\frac{\log 25}{\log \sqrt{5}}$
c) $\frac{\log 243}{\log 27}$
d) $\frac{\log 0{,}5}{\log 4}$

Solutions

a) $\frac{\log 16}{\log 4}$, firstly we have to express 16 and 4 as powers of the same number

$= \frac{\log 4^2}{\log 4}$, dropping the powers in both numerator and denominator we get;

$= \frac{2\log 4}{1\log 4}$, log 4 in the numerator and denominator cancels

$= \frac{2}{1}$

$\therefore \frac{\log 16}{\log 4} = 2$

b) $\frac{\log 25}{\log \sqrt{5}}$

$= \frac{\log 5^2}{\log 5^{\frac{1}{2}}}$

$= \frac{2\log 5}{\frac{1}{2}\log 5}$

$= 2 \div \frac{1}{2}$

$= 2 \times 2,$

$\therefore \frac{\log 25}{\log \sqrt{5}} = 4$

c) $\frac{\log 243}{\log 27}$

$= \frac{\log 3^5}{\log 3^3}$, dropping the powers we get;

$= \frac{5\log 3}{3\log 3}$

$= \frac{5}{3}$

$= 1\frac{2}{3}$

d) $\frac{\log 0{,}5}{\log 4}$

$= \frac{\log \frac{1}{2}}{\log 4}$

$= \frac{\log 2^{-1}}{\log 2^2}$

$= \frac{-1\ \log 2}{2\log 2}$

$= -\frac{1}{2}$

Worked Example [16]

Questions

Evalute the following logarithms

a) $\frac{\log 9 - \log 3}{\log 27 - \log 9}$

b) $\frac{\log 4 + \log 2}{\log 32 - \log 2}$

c) $\frac{\log 2 - \log 3}{\log 8 - \log 27}$

d) $\frac{\log 25 - \log 16}{\log 5 - \log 4}$

Solution

a) $\frac{\log 9 - \log 3}{\log 27 - \log 9}$

$= \frac{\log\ (\frac{9}{3})}{\log\ (\frac{27}{9})}$

$= \frac{\log 3}{\log 3}$

$=1$

b) $\frac{\log 4 + \log 2}{\log 32 - \log 2}$

$= \frac{\log\ (4\times 2)}{\log \frac{32}{2}}$

$= \frac{\log 8}{\log 16}$

$= \frac{\log 2^3}{\log 2^4}$

$= \frac{3 \log 2}{4 \log 2}$

$= \frac{3}{4}$

c) $\frac{\log 2 - \log 3}{\log 8 - \log 27}$

$= \frac{\log \frac{2}{3}}{\log \frac{8}{27}}$

$= \frac{\log \left(\frac{2}{3}\right)}{\log\ \left(\frac{2}{3}\right)^3}$

$= \frac{1 \log\ \left(\frac{2}{3}\right)}{3 \log\ \left(\frac{2}{3}\right)}$

$= \frac{1}{3}$

d) $\frac{\log 25 - \log 16}{\log 5 - \log 4}$

$= \frac{\log \left(\frac{25}{16}\right)}{\log \left(\frac{5}{4}\right)}$

$= \frac{\log \left(\frac{5}{4}\right)^2}{\log \left(\frac{5}{4}\right)}$

$= \frac{2 \log\ \left(\frac{5}{4}\right)}{1 \log \left(\frac{5}{4}\right)}$

$= 2$

Worked Example [17]

Questions

Evaluate the following logarithms

a) $\log_2 32$

b) $\log_3 \sqrt{3}$

c) $\log_4 0{,}25$

d) $\log_{0,2} 5$

Solutions

There are 2 methods which can be used to solve these logarithms

a) <u>Method 1</u>

you let $2^x = 32$, then solving the equation ;

$2^x = 2^5$

$\therefore x = 5.$

$\therefore \log_2 32 = 5$

<u>Method 2</u>

$\log_2 32$

$= \log_2 2^5$ dropping the power we get;

$= 5\log_2 2$, but $\log_2 2 = 1$

$= 5 \times 1$

$= 5.$

b) <u>Using Method 1</u>

let$3^x = \sqrt{3}$

$3^x = 3^{\frac{1}{2}}$

$x = \frac{1}{2}$

$\therefore \ \log_3 \sqrt{3} = \frac{1}{2}$

<u>Using Method 2</u>

Try to use method 2 and get the same answer

c) **<u>Using Method 1</u>**

Try to use method 2 and get the same answer

<u>Using Method 2</u>

$\log_4 0{,}25$

$= \log_4 \frac{25}{100}$

$= \log_4(\frac{1}{4})$

$=\log_4 4^{-1}$, dropping the power we get;

$=-1\log_4 4, \quad$ but $\log_4 4 = 1$

$= -1$

d) **<u>Using Method 1</u>**

$\log_{0,2} 5$

let$0{,}2^{\mathrm{x}} = 5$

$\left(\frac{2}{10}\right)^{\mathrm{x}} = 5,$

$(\frac{1}{5})^{\mathrm{x}} = 5$, expressing $\frac{1}{5}$ as powers of 5

$(5^{-1})^{\mathrm{x}} = 5,$

$5^{-\mathrm{x}} = 5,$

$\mathrm{x} = -1,$

$\therefore \log_{0,2} 5 = -1$

<u>Using Method 2</u>

Try to use Method 2 and get the same answer.

Worked Example [18]

Questions

Simplify as far as possible

a) $\log 1\frac{1}{2} - \log 3 + \log 2$

b) $\log 2{,}7 + 2 - 3\log 3$

c) $3\log_{10} 3 + \log_{10} 16 - \log_{10} 36$

Solutions

a) Applying the laws of logarithms

$\log 1\frac{1}{2} - \log 3 + \log 2$

$= \log\ (1\frac{1}{2} \div 3 \times 2)$

$= \log\ (\frac{3}{2} \times \frac{1}{3} \times 2)$, simplifying we get

$= \log 1$

$= 0$

b) $\log 2{,}7 + 2 - 3\log 3$

$= \log 2{,}7 + \log 100 - \log 3^3$

$= \log(2{,}7 \times 100 \div 3^3)$

$= \log\left(\frac{2{,}7\times 100}{27}\right)$

$= \log_{10} 10$

$= 1$

c) $3\log_{10} 3 + \log_{10} 16 - \log_{10} 36$

$= \log_{10}(3^3 \times 16 \div 36)$

$= \log_{10}(\frac{27\times 16}{36})$

$= \log_{10} 12$

Activity (*15.4*) Application of theories of logarithms

Questions

1) Simplify the following

a) $\frac{\log 81}{\log\sqrt{9}}$

b) $\frac{\log\frac{1}{3}}{\log 27}$

c) $\frac{\log(\frac{1}{64})}{\log 0{,}25}$

2) Evaluate the following

a) $\frac{\log 3-\log 7}{\log 27-\log 343}$

b) $\frac{\log 64-\log 16}{\log 32-\log 4}$

c) $\frac{\log 27+\log 3}{\log 9-\log 27}$

3) Evaluate the following

a) $\log_{0,2} 125$

b) $\log_{\sqrt{6}}(\frac{1}{36})$

c) $\log_8 0{,}125$

4) Simplify as far as possible

a) $\log 1{,}6 + 3 - 2\log 4$

b) $3\log_{10} 2 + 3\log_{10} 5 - 1$

c) $\log_4 64 + \log_4 16 - \log_4 256$

Answers

1 a) 4	b) $\frac{1}{9}$	c) 3
2 a) $\frac{1}{3}$	b) $\frac{2}{3}$	c) -4
3 a) -3	b) -4	c) -1
4 a) 3	b) 2	c) 1

15.2.5 Equations involving logarithms

Worked Example [19]

Question

Solve the equation $\log_{10} x = 2$

Solution

First we rewrite the equation in index form

$\log_{10} x = 2$

$10^2 = x$

$\therefore x = 100$

Worked Example [20]

Questions

Solve for x in $\log_x 8 = 1{,}5$

Solution

$\log_x 8 = 1{,}5$

$x^{1,5} = 8$

$x^{\frac{3}{2}} = 8$

$x = 8^{\frac{2}{3}}$

$x = 4$

Worked Example [21]

Question

Solve the equation $\log 3x + \log 4 = 1$

Solution

Rewrite the equation in logarithmic form

$\log 3x + \log 4 = \log 10$, applying laws of logarithms we get;

$\log(3x \times 4) = \log 10$

$\log 12x = \log 10$

$12x = 10$, solving the equation

$\therefore x = \dfrac{5}{6}$

Worked Example [22]

Question

Given that $\log_5 2x - \log_5(5x + 1) = 0$, find the value of x.

Solution

Applying the laws of logarithms we get

$\log_5(\frac{2x}{5x+1}) = 0$, but $0 = \log_5 1$

$\log_5\left(\frac{2x}{5x+1}\right) = \log_5 1$

$\frac{2x}{5x+1} = 1$, clearing the denominator

$2x = 5x + 1$

$3x = -1$

$\therefore x = -\frac{1}{3}$

Activity (*15.5*) Equations involving logarithms

Questions

Solve the following equations

1) $\log x - \log(x+2) = 1$
2) $\log_3 x = -1$
3) $\log_2 x - \log_2(2x-1) = 2$
4) $\log_x 9 = -\frac{1}{2}$
5) Given that $\log x + \log(3+x) = 1$ form an equation in x and show that it reduces to

   $x^2 + 3x - 10 = 0$

   hence or otherwise find the value of x.

Answers

1) $-2\frac{2}{9}$
2) $\frac{1}{3}$
3) $\frac{4}{7}$
4) $\frac{1}{81}$
5) -5 or 2

REFLECTION

- Indices and logarithms work hand in hand , the have a unique relationship as seen in the unit
- Laws involving indices have a relationship with theory of logarithms.

15.4 SUMMARY

This unit covered Laws of indices and 3 Theory of logarithms. The unit also looked at the two types of equations involving indices. There is need to make use of these laws when simplifying indices or logarithms.

15.5 Further Reading

1. Macrae, M, Madungwe, L and Mutangadura, A (2017) New General Mathematics Book 3, Pearson Capetown.
2. Macral, M, Madungwe, L and Mutangadura, A (2017) New General Mathematics Book 4, Pearson Capetown.
3. Meyers, C; Graham, B; Dawe, L (2004) Mathscape Working Mathematically, 9th Ed, MacMillan
4. Pimentel R and Wall T (2011) International Mathematics, Hodder UK
5. Rayner D (2005) Extended Mathematics. Oxford New York

15.6 ASSESSMENT TEST

1) It is given that $\log_x 81 = \log_4 64$, find the value of x
2) Simplify as far as possible $2\log_{10} 5 + 2 - \log 2{,}5$
3) Given that $\log 2 = 0{,}3010$ and $\log 3 = 0{,}477$, find
   a) $\log 60$ b) $\log 1.5$ c) $\log\sqrt{2}$
4) Evaluate the following
   a) $(8x^3)^{\frac{2}{3}}$
   b) $(0{,}04)^{-\frac{3}{2}}$
   c) $(\frac{1}{9})^{-\frac{1}{2}}$
5) Solve for x in $4x^4 = 324$